# Final Examination (Spring 2010)

(Answers should be either in English or Korean.)

(1) For $\mathbf{A} \in \mathbb{R}^{m\times n}$, the singular value decomposition of **A** results in $\mathbf{A} = \mathbf{U\Sigma V}^T$ with $\mathbf{\Sigma} = \mathrm{diag}(\sigma_1, \cdots, \sigma_r)$, $\mathbf{U} = [\mathbf{u}_1 \cdots \mathbf{u}_r] \in \mathbb{R}^{m\times r}$, $\mathbf{V} = [\mathbf{v}_1 \cdots \mathbf{v}_r] \in \mathbb{R}^{n\times r}$, $\mathbf{U}^T\mathbf{U} = \mathbf{I}_r$, and $\mathbf{V}^T\mathbf{V} = \mathbf{I}_r$.

(a) Show that $\mathbf{v}_i$ is an eigenvector of $\mathbf{A}^T\mathbf{A}$ and find its eigenvalue.
(b) Show that $\mathbf{u}_i$ is an eigenvector of $\mathbf{A}\mathbf{A}^T$ and find its eigenvalue.
(c) Show that $\mathbf{A}\mathbf{v}_i = \sigma_i\mathbf{u}_i$.
(d) Find the pseudo-inverse of **A**.
(e) For $\hat{\mathbf{A}} = \sum_{i=1}^{p} \sigma_i \mathbf{u}_i \mathbf{v}_i$ with $p < r$, find $\left\| \mathbf{A} - \hat{\mathbf{A}} \right\|$.

(2) You are solving the following minimization problem using the steepest descent method: $(x^*, y^*) = \arg\min_{(x,y)} \Phi(x,y) = \arg\min_{(x,y)} \left\{ \left(x-\sqrt{2}\right)^2 + \left(y-\sqrt{2}\right)^2 \right\}$. Your initial guess is $(x_0, y_0) = (0,0)$ and the step size is $h_k = 1/2^{k-1}$ where $k$ is the iteration number. Find $(x_k, y_k)$ and $\Phi(x_k, y_k)$ at the first ($k$=1), second ($k$=2), and third ($k$=3) iteration steps. Plot the search trajectory.

(3) You are solving the following minimization problem using the Newton method: $(x^*, y^*) = \arg\min_{(x,y)} \Phi(x,y) = \arg\min_{(x,y)} \left\{ \frac{\left(x-\sqrt{2}\right)^2}{25} + \left(y-\sqrt{2}\right)^2 \right\}$. Your initial guess is $(x_0, y_0) = (0,0)$ and the step size is $h_k = 1/2^{k-1}$ where $k$ is the iteration number. Find $(x_k, y_k)$ and $\Phi(x_k, y_k)$ at the first ($k$=1), second ($k$=2), and third ($k$=3) iteration steps. Plot the search trajectory.

(4) Construct an iterative algorithm which minimizes the following nonlinear objective function using the Gauss-Newton method with the Tikhonov regularization: $\mathbf{x}^* = \arg\min_{\mathbf{x}} \|\mathbf{r}\|^2 = \arg\min_{\mathbf{x}} \{\mathbf{f}(\mathbf{x}) - \mathbf{y}\}^T \{\mathbf{f}(\mathbf{x}) - \mathbf{y}\}$ where $\mathbf{x} \in \mathbb{R}^P$, $\mathbf{f}(\mathbf{x}) \in \mathbb{R}^M$, and $\mathbf{y} \in \mathbb{R}^M$.

(5) You are provided with a 16-channel EIT system using the neighboring data collection method. Construct a linear time-difference conductivity image reconstruction algorithm using an inverse mesh with 100 elements (pixels) and the truncated singular value decomposition of a sensitivity matrix.