Final Exam (Fall 2011)

(1) Derive two fundamental equations in impedance imaging.

(a) From Maxwell's equations of $\nabla \times \mathbf{E} = -j\omega\mu_0\mathbf{H}$ and $\nabla \times \mathbf{H} = \tau\mathbf{E} = (\sigma + j\omega\varepsilon)\mathbf{E} = \mathbf{J}$, derive the first fundamental equation in impedance imaging, which is expressed by **H** and τ. Provide your interpretation of the derived equation.

(b) Using that fact that $\nabla \cdot (\mu_0\mathbf{H}) = 0$, define a vector magnetic potential **A**. From $\nabla \times (\mathbf{E} + j\omega\mathbf{A}) = 0$, define a scalar potential u. Noting that $\nabla \cdot \mathbf{J} = 0$, derive the second fundamental equation in impedance imaging, which is expressed by u, **A** and τ. Provide your interpretation of the derived equation.

(2) Consider an electrically conducting homogeneous cylindrical domain with a known conductivity σ=1 S/m. You attached two electrodes with the same surface area of A at the middle positions of the cylinder on the left and right sides and injected dc current of I mA.

(a) Denoting the voltage in the domain Ω by u, describe a partial differential equation with proper boundary conditions.

(b) On the cross-sectional plane at the center of the cylinder, draw current streamlines.

(c) On the same cross-sectional plane, draw equipotential lines.

(d) Derive expressions for the current density **J** and magnetic flux density **B**.

(3) You are provided with a 16-channel EIT system using the neighboring data collection method.

(a) Define a projection to collect the first set of boundary voltage data subject to the current injection between the first neighboring electrode pair.

(b) Define the full set of boundary voltage data subject to 16 injection currents.

(c) Derive the principle of the reciprocity and calculate the number of independent voltage data.

(d) Assume that you designed an image reconstruction mesh with 100 elements (pixels). Derive an expression of the sensitivity matrix. Provide interpretations for each row and column of the sensitivity matrix.

(e) Construct a linear time-difference conductivity image reconstruction algorithm using the truncated singular value decomposition of the sensitivity matrix.

(f) Explain how to determine the rank of the sensitivity matrix and discuss its implication to the spatial resolution.

(4) Inside an MRI scanner, you are conducting an MREIT imaging experiment using two pairs of surface electrodes.

(a) For current injection along the horizontal pair of the electrodes, draw a spin-echo based pulse sequence to inject positive and negative current pulses. Explain why you need to inject both positive and negative currents.

(b) Derive equations to extract the *z*-component B_z of the induced magnetic flux density $\mathbf{B}$=(B_x, B_y, B_z) from the following *k*-space data:

$$S^{\pm}(m,n)=\iint M(x,y)e^{j\delta(x,y)}e^{\pm j\gamma B_z(x,y)T_c}e^{-j(xm\Delta k_x+yn\Delta k_y)}dxdy\ .$$

(c) Using the *z*-component of the fundamental relation of $\nabla^2\mathbf{B}=-\mu_0\nabla u\times\nabla\sigma$ where u is the voltage and σ is the conductivity at the position $\mathbf{r}$, derive the core equation between the two-dimensional gradient of σ and B_z. Assume that you know the voltage u.

(d) Using two data sets of B_{z1} and B_{z2} subject to two injection currents, derive the core equation of the harmonic B_z algorithm in a matrix-vector form.

(e) In practice, you do not know the voltage u since it is a nonlinear function of the unknown conductivity σ. Suggest an iterative method to recover the conductivity σ.