

## Finite Element Method (FEM)

### (A) Electrostatic problem

- $\nabla \cdot \mathbf{D} = \rho$ ,  $\mathbf{E} = \varepsilon \mathbf{D} = -\nabla u \Rightarrow \nabla \cdot \nabla u = \nabla^2 u = -\frac{\rho}{\varepsilon}$
- $\nabla \cdot \mathbf{J} = 0$ ,  $\mathbf{J} = \sigma \mathbf{E} = -\sigma \nabla u \Rightarrow \nabla \cdot (\sigma \nabla u) = 0$

### (B) One-dimensional problem

- Governing equation (differential equation):  $-\frac{d}{dx} \left( \sigma(x) \frac{du(x)}{dx} \right) = f(x)$
- Domain:  $x_a \leq x \leq x_b$
- Essential boundary condition (Dirichlet boundary condition):  $u(x_a) = u_a$
- Natural boundary condition (Neumann boundary condition):  $-\sigma(x) \frac{du(x)}{dx} \Big|_{x=x_b} = j_b$

#### • Trial solution

- $\tilde{u}(x; a) = \phi_0(x) + a_1 \phi_1(x) + \dots + a_N \phi_N(x) = \phi_0(x) + \sum_{j=1}^N a_j \phi_j(x)$
- $a_i$  are values of  $u(x)$  at some points
- We usually use polynomials for  $\phi_i(x)$

#### • Residual for governing equation

- Residual:  $R(x; a) = -\frac{d}{dx} \left( \sigma(x) \frac{d\tilde{u}(x)}{dx} \right) - f(x)$
- Orthogonality principle (Galerkin method):

$$\langle R(x; a), \phi_i(x) \rangle = \int_{x_a}^{x_b} R(x; a) \phi_i(x) dx = 0, \quad i = 1, \dots, N$$

$$\Rightarrow \int_{x_a}^{x_b} \left\{ -\frac{d}{dx} \left( \sigma(x) \frac{d\tilde{u}(x)}{dx} \right) - f(x) \right\} \phi_i(x) dx = 0$$

$$\Rightarrow \int_{x_a}^{x_b} \frac{d}{dx} \left( \sigma(x) \frac{d\tilde{u}(x)}{dx} \right) \phi_i(x) dx = -\int_{x_a}^{x_b} f(x) \phi_i(x) dx$$

- **Integration by part**

$$\int_{x_a}^{x_b} \frac{d}{dx}(fg)dx = [gf]_{x_a}^{x_b} = \int_{x_a}^{x_b} \frac{df}{dx}gdx + \int_{x_a}^{x_b} \frac{dg}{dx}fdx \Rightarrow \int_{x_a}^{x_b} \frac{df}{dx}gdx = [gf]_{x_a}^{x_b} - \int_{x_a}^{x_b} \frac{dg}{dx}fdx$$

$$\int_{x_a}^{x_b} \frac{d}{dx} \left( \sigma(x) \frac{d\tilde{u}(x)}{dx} \right) \phi_i(x) dx = - \left[ \left( -\sigma(x) \frac{d\tilde{u}(x)}{dx} \right) \phi_i(x) \right]_{x_a}^{x_b} - \int_{x_a}^{x_b} \sigma(x) \frac{d\tilde{u}(x)}{dx} \frac{d\phi_i(x)}{dx} dx$$

- **System equations**

$$\int_{x_a}^{x_b} \sigma(x) \frac{d\tilde{u}(x)}{dx} \frac{d\phi_i(x)}{dx} dx = - \left[ \left( -\sigma(x) \frac{d\tilde{u}(x)}{dx} \right) \phi_i(x) \right]_{x_a}^{x_b} + \int_{x_a}^{x_b} f(x) \phi_i(x) dx$$

$$\frac{d\tilde{u}}{dx} = \frac{d\phi_0}{dx} + \sum_{j=1}^N a_j \frac{d\phi_j}{dx}$$

$$\begin{aligned} & \int_{x_a}^{x_b} \sigma(x) \frac{d\tilde{u}(x)}{dx} \frac{d\phi_i(x)}{dx} dx \\ &= \int_{x_a}^{x_b} \sigma(x) \left\{ \frac{d\phi_0}{dx} + \sum_{j=1}^N a_j \frac{d\phi_j}{dx} \right\} \frac{d\phi_i(x)}{dx} dx \\ &= \sum_{j=1}^N \left\{ \int_{x_a}^{x_b} \sigma(x) \frac{d\phi_i(x)}{dx} \frac{d\phi_j(x)}{dx} dx \right\} a_j + \int_{x_a}^{x_b} \sigma(x) \frac{d\phi_i(x)}{dx} \frac{d\phi_0(x)}{dx} dx \end{aligned}$$

$$\begin{aligned} & \sum_{j=1}^N \left\{ \int_{x_a}^{x_b} \sigma(x) \frac{d\phi_i(x)}{dx} \frac{d\phi_j(x)}{dx} dx \right\} a_j \\ &= - \left[ \left( -\sigma(x) \frac{d\tilde{u}(x)}{dx} \right) \phi_i(x) \right]_{x_a}^{x_b} + \int_{x_a}^{x_b} f(x) \phi_i(x) dx - \int_{x_a}^{x_b} \sigma(x) \frac{d\phi_i(x)}{dx} \frac{d\phi_0(x)}{dx} dx \end{aligned}, \quad i=1, \dots, N$$

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$$\begin{aligned} & \left( \int_{x_a}^{x_b} \sigma(x) \frac{d\phi_1(x)}{dx} \frac{d\phi_1(x)}{dx} dx \right) a_1 + \dots + \left( \int_{x_a}^{x_b} \sigma(x) \frac{d\phi_1(x)}{dx} \frac{d\phi_N(x)}{dx} dx \right) a_N \\ & = - \left[ \left( -\sigma(x) \frac{d\tilde{u}(x)}{dx} \right) \phi_1(x) \right]_{x_a}^{x_b} + \int_{x_a}^{x_b} f(x) \phi_1(x) dx - \int_{x_a}^{x_b} \sigma(x) \frac{d\phi_1(x)}{dx} \frac{d\phi_0(x)}{dx} dx \end{aligned} \quad (i = 1)$$

⋮

$$\begin{aligned} & \left( \int_{x_a}^{x_b} \sigma(x) \frac{d\phi_N(x)}{dx} \frac{d\phi_1(x)}{dx} dx \right) a_1 + \dots + \left( \int_{x_a}^{x_b} \sigma(x) \frac{d\phi_N(x)}{dx} \frac{d\phi_N(x)}{dx} dx \right) a_N \\ & = - \left[ \left( -\sigma(x) \frac{d\tilde{u}(x)}{dx} \right) \phi_N(x) \right]_{x_a}^{x_b} + \int_{x_a}^{x_b} f(x) \phi_N(x) dx - \int_{x_a}^{x_b} \sigma(x) \frac{d\phi_N(x)}{dx} \frac{d\phi_0(x)}{dx} dx \end{aligned} \quad (i = N)$$

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$$[K]\{a\} = \{F\}$$

$$K_{ij} = \int_{x_a}^{x_b} \sigma(x) \frac{d\phi_i(x)}{dx} \frac{d\phi_j(x)}{dx} dx \text{ and}$$

$$F_i = - \left[ \left( -\sigma(x) \frac{d\tilde{u}(x)}{dx} \right) \phi_i(x) \right]_{x_a}^{x_b} + \int_{x_a}^{x_b} f(x) \phi_i(x) dx - \int_{x_a}^{x_b} \sigma(x) \frac{d\phi_i(x)}{dx} \frac{d\phi_0(x)}{dx} dx$$

• **One linear element with two nodes**

- Find  $\phi_i(x)$

(1) Let's assume  $\tilde{u}(x; \alpha) = \alpha_1 + \alpha_2 x$  on the domain  $x_a \leq x \leq x_b$

(2) At each node,  $\tilde{u}(x_a; \alpha) = \alpha_1 + \alpha_2 x_a = a_1$  and  $\tilde{u}(x_b; \alpha) = \alpha_1 + \alpha_2 x_b = a_2$

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$$\begin{bmatrix} 1 & x_a \\ 1 & x_b \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

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$$\alpha_1 = \frac{x_b a_1 - x_a a_2}{x_b - x_a} \quad \text{and} \quad \alpha_2 = \frac{a_2 - a_1}{x_b - x_a}$$

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$$\tilde{u}(x; a) = \frac{x_b a_1 - x_a a_2}{x_b - x_a} + \frac{a_2 - a_1}{x_b - x_a} x = a_1 \frac{x_b - x}{x_b - x_a} + a_2 \frac{x - x_a}{x_b - x_a} = a_1 \phi_1 + a_2 \phi_2$$

$$\phi_1(x) = \frac{x_b - x}{x_b - x_a} \quad \text{and} \quad \phi_2(x) = \frac{x - x_a}{x_b - x_a} : \text{ interpolation functions}$$

$$\phi_1(x_a) = 1, \quad \phi_1(x_b) = 0 \quad \text{and} \quad \phi_2(x_a) = 0, \quad \phi_2(x_b) = 1 \Rightarrow \phi_j(x_i) = \delta_{ji}$$

(3) Other method for finding  $\phi_i(x)$  from interpolation property

Let  $\tilde{u}(x; a) = a_1 \phi_1(x) + a_2 \phi_2(x)$ ,  $\phi_1(x) = \beta_1 + \beta_2 x$ , and  $\phi_2(x) = \gamma_1 + \gamma_2 x$

From  $\phi_1(x_a) = 1$ ,  $\phi_1(x_b) = 0$  and  $\phi_2(x_a) = 0$ ,  $\phi_2(x_b) = 1$ , we get

$$\phi_1(x) = \frac{x_b - x}{x_b - x_a} \quad \text{and} \quad \phi_2(x) = \frac{x - x_a}{x_b - x_a}$$

- Evaluate system equations

(1) If  $f$  is not zero,

$$[K]\{a\} = \{F\}$$

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$$\begin{bmatrix} \frac{1}{2} \frac{x_b + x_a}{x_b - x_a} & -\frac{1}{2} \frac{x_b + x_a}{x_b - x_a} \\ -\frac{1}{2} \frac{x_b + x_a}{x_b - x_a} & \frac{1}{2} \frac{x_b + x_a}{x_b - x_a} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -\left[ \left( -\sigma(x) \frac{d\tilde{u}(x)}{dx} \right) \phi_1(x) \right]_{x_a}^{x_b} + \int_{x_a}^{x_b} f(x) \phi_1(x) dx \\ -\left[ \left( -\sigma(x) \frac{d\tilde{u}(x)}{dx} \right) \phi_2(x) \right]_{x_a}^{x_b} + \int_{x_a}^{x_b} f(x) \phi_2(x) dx \end{bmatrix}$$

(2) If  $f$  is zero,

$$[K]\{a\} = \{F\}$$

⇓

$$\begin{bmatrix} \frac{1}{2} \frac{x_b + x_a}{x_b - x_a} & -\frac{1}{2} \frac{x_b + x_a}{x_b - x_a} \\ -\frac{1}{2} \frac{x_b + x_a}{x_b - x_a} & \frac{1}{2} \frac{x_b + x_a}{x_b - x_a} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -\left[ \left( -\sigma(x) \frac{d\tilde{u}(x)}{dx} \right) \phi_1(x) \right]_{x_a}^{x_b} \\ -\left[ \left( -\sigma(x) \frac{d\tilde{u}(x)}{dx} \right) \phi_2(x) \right]_{x_a}^{x_b} \end{bmatrix} = \begin{bmatrix} \left( -\sigma(x) \frac{d\tilde{u}(x)}{dx} \right)_{x_a} \\ -\left( -\sigma(x) \frac{d\tilde{u}(x)}{dx} \right)_{x_b} \end{bmatrix} = \begin{bmatrix} j_a \\ -j_b \end{bmatrix}$$

with  $j_a + (-j_b) = 0$

• Many linear elements (each element with two nodes)

- For  $k$ -th element, evaluate  $[K^{(k)}] \{a^{(k)}\} = \{F^{(k)}\}$  with  $k = 1, \dots, E$

- Assemble system equations as

$$\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 & 0 & \dots & \dots & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} + K_{11}^{(2)} & K_{12}^{(2)} & 0 & 0 & \dots & \dots & 0 \\ 0 & K_{21}^{(2)} & K_{22}^{(2)} + K_{11}^{(3)} & K_{12}^{(3)} & 0 & \dots & \dots & 0 \\ 0 & 0 & K_{21}^{(3)} & K_{22}^{(3)} + K_{11}^{(4)} & K_{12}^{(4)} & \dots & \dots & 0 \\ 0 & 0 & 0 & K_{21}^{(4)} & K_{22}^{(4)} + K_{11}^{(5)} & \dots & \dots & \vdots \\ 0 & 0 & 0 & 0 & K_{21}^{(5)} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & K_{22}^{(E-1)} + K_{11}^{(E)} & K_{12}^{(E)} \\ 0 & 0 & 0 & \dots & \dots & 0 & K_{21}^{(E)} & K_{22}^{(E)} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ \vdots \\ a_{E-1} \\ a_E \end{bmatrix} = \begin{bmatrix} F_1^{(1)} \\ F_2^{(1)} + F_1^{(2)} \\ F_2^{(2)} + F_1^{(3)} \\ F_2^{(3)} + F_1^{(4)} \\ F_2^{(5)} + F_1^{(6)} \\ \vdots \\ F_2^{(E-1)} + F_1^{(E)} \\ F_2^{(E)} \end{bmatrix}$$

using local node numbers, or

$$\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 & 0 & \dots & \dots & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} + K_{22}^{(2)} & K_{23}^{(2)} & 0 & 0 & \dots & \dots & 0 \\ 0 & K_{32}^{(2)} & K_{33}^{(2)} + K_{33}^{(3)} & K_{34}^{(3)} & 0 & \dots & \dots & 0 \\ 0 & 0 & K_{43}^{(3)} & K_{44}^{(3)} + K_{44}^{(4)} & K_{45}^{(4)} & \dots & \dots & 0 \\ 0 & 0 & 0 & K_{54}^{(4)} & K_{55}^{(4)} + K_{55}^{(5)} & \dots & \dots & \vdots \\ 0 & 0 & 0 & 0 & K_{65}^{(5)} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & K_{(E-1)(E-1)}^{(E-1)} + K_{(E-1)(E-1)}^{(E)} & K_{(E-1)E}^{(E)} \\ 0 & 0 & 0 & \dots & \dots & 0 & K_{E(E-1)}^{(E)} & K_{EE}^{(E)} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ \vdots \\ a_{E-1} \\ a_E \end{bmatrix} = \begin{bmatrix} F_1^{(1)} \\ F_2^{(1)} + F_2^{(2)} \\ F_3^{(2)} + F_3^{(3)} \\ F_4^{(3)} + F_4^{(4)} \\ F_5^{(5)} + F_5^{(6)} \\ \vdots \\ F_{E-1}^{(E-1)} + F_{E-1}^{(E)} \\ F_2^{(E)} \end{bmatrix}$$

using global node numbers

• One-dimensional  $C^0$ -quadratic isoparametric element

- Define a master or parent element along  $\xi$  – axis with three nodes uniform spaced at  $\xi = -1, 0, +1$
- Define  $C^0$ -quadratic parent shape functions,  $\phi_i(\xi)$  as

$$\phi_1(\xi) = \frac{(\xi - \xi_2)(\xi - \xi_3)}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)} = \frac{\xi(\xi - 1)}{2},$$

$$\phi_2(\xi) = \frac{(\xi - \xi_1)(\xi - \xi_3)}{(\xi_2 - \xi_1)(\xi_2 - \xi_3)} = -(\xi + 1)(\xi - 1),$$

$$\phi_3(\xi) = \frac{(\xi - \xi_1)(\xi - \xi_2)}{(\xi_3 - \xi_1)(\xi_3 - \xi_2)} = \frac{(\xi + 1)\xi}{2}$$

- Define a coordinate transformation,  $x = \chi^{(e)}(\xi)$  for each element ( $e$ ) using the isoparametric approach as

$$x = \sum_{k=1}^L x_k^{(e)} \psi_k(\xi)$$

where  $x_k^{(e)}$ ,  $k=1, \dots, L$  are coordinates of  $L$  nodes in the element ( $e$ ) and  $\psi_k(\xi) = \phi_k(\xi)$  for  $k=1, \dots, L$  (therefore, isoparametric). Therefore,

$$\begin{aligned} x &= \sum_{k=1}^3 x_k^{(e)} \phi_k(\xi) \\ &= \frac{1}{2} \xi(\xi - 1)x_1^{(e)} + (1 + \xi)(1 - \xi)x_2^{(e)} + \frac{1}{2} \xi(\xi + 1)x_3^{(e)} \\ &(\xi_1 = -1 \Leftrightarrow x_1^{(e)}, \xi_2 = 0 \Leftrightarrow x_2^{(e)}, \xi_3 = 1 \Leftrightarrow x_3^{(e)}) \end{aligned}$$

- Evaluate element matrix,  $[K^{(e)}]$  as

$$K_{ij}^{(e)} = \int_x^{x_3} \frac{d\phi_i^{(e)}(x)}{dx} \sigma(x) \frac{d\phi_j^{(e)}(x)}{dx} dx$$

where

$$\frac{d\phi_i^{(e)}(x)}{dx} = \frac{d\phi_i^{(e)}(x)}{d\xi} \frac{d\xi}{dx} = \frac{1}{J^{(e)}(\xi)} \frac{d\phi_i(\xi)}{d\xi} \quad \text{with} \quad J^{(e)}(\xi) = \frac{dx}{d\xi}$$

Note that  $\phi_i^{(e)}(x) = \phi_i^{(e)}(\chi^{(e)}(\xi)) = \phi_i(\xi)$  and  $\alpha(x) = \alpha(\chi^{(e)}(\xi)) = \alpha^{(e)}(\xi)$ . Therefore,

$$K_{ij}^{(e)} = \int_{-1}^{+1} \left( \frac{1}{J^{(e)}(\xi)} \frac{d\phi_i(\xi)}{d\xi} \right) \sigma^{(e)}(\xi) \left( \frac{1}{J^{(e)}(\xi)} \frac{d\phi_j(\xi)}{d\xi} \right) J^{(e)}(\xi) d\xi$$

Similarly,

$$Ff_i^{(e)} = \int_{-1}^{+1} f^{(e)}(\xi) \phi_i(\xi) J^{(e)}(\xi) d\xi$$

- Numerical integration

$$\int_{-1}^{+1} I(\xi) d\xi \cong \sum_{l=1}^N w_{nl} I(\xi_{nl}) \quad \text{see Table 8.12 Gauss-Legendre integration (Burnett)}$$

### (C) Two-dimensional problem

- Governing equation (differential equation):

$$-\frac{\partial}{\partial x} \left( \sigma(x, y) \frac{\partial u(x, y)}{\partial x} \right) - \frac{\partial}{\partial y} \left( \sigma(x, y) \frac{\partial u(x, y)}{\partial y} \right) = f(x, y)$$

- Domain:  $s = (x, y) \in \Omega$

- Boundary:  $\partial\Omega$

- Outward unit normal vector:  $\{n\} = \begin{Bmatrix} n_x \\ n_y \end{Bmatrix}$

- Essential boundary condition (Dirichlet boundary condition):  $u(s_a) = u_a, s_a \in \partial\Omega$

- Natural boundary condition (Neumann boundary condition):

$$-\sigma(x, y) \nabla u(x, y) \cdot \{n\} \Big|_{s_b} = j_b, s_b \in \partial\Omega$$

- Material property

(1) Homogeneous and isotropic on  $\Omega$ :  $\sigma(x, y) = \sigma$

(2) Homogeneous and isotropic within element:  $\sigma^{(e)}(x, y) = \sigma^{(e)}$

(3) Anisotropic:  $[\sigma] = \begin{bmatrix} \sigma_x & 0 \\ 0 & \sigma_y \end{bmatrix}$

#### • Element equation

- Following the same procedure as in one-dimensional problem with

$$\tilde{u}^{(e)}(x, y; a) = \sum_{j=1}^N a_j \phi_j^{(e)}(x, y),$$

$$[K]\{a\} = \{F\}$$

where

$$K_{ij}^{(e)} = \iint_{(e)} \frac{\partial \phi_i^{(e)}}{\partial x} \sigma_x \frac{\partial \phi_j^{(e)}}{\partial x} dx dy + \iint_{(e)} \frac{\partial \phi_i^{(e)}}{\partial y} \sigma_y \frac{\partial \phi_j^{(e)}}{\partial y} dx dy$$

$$F_i^{(e)} = \iint_{(e)} f \phi_i^{(e)} dx dy - \oint_{(e)} \left( -\sigma_x \frac{\partial \tilde{u}^{(e)}}{\partial x} n_x^{(e)} - \sigma_y \frac{\partial \tilde{u}^{(e)}}{\partial y} n_y^{(e)} \right) dl$$

• **C<sup>0</sup>-linear triangular element**

-  $\tilde{u}^{(e)}(x, y) = \alpha + \beta x + \gamma y = \sum_{i=1}^3 a_i \phi_i^{(e)}(x, y)$  and  $\tilde{u}^{(e)}(x_i, y_i) = a_i$  with  $i = 1, \dots, 3$

- Find  $\alpha, \beta, \gamma$  from

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Then,  $\phi_i^{(e)}(x, y) = \frac{\alpha_i + \beta_i x + \gamma_i y}{2\Delta}$  with

$$\alpha_i = x_j y_k - x_k y_j, \quad \beta_i = y_k - y_j, \quad \gamma_i = x_k - x_j, \quad \Delta = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \text{area of element}$$

- Element matrix

$$K_{ij} = \frac{\sigma_x^{(e)}}{4\Delta} \beta_i \beta_j + \frac{\sigma_y^{(e)}}{4\Delta} \gamma_i \gamma_j$$

$$F f_i^{(e)} = \frac{f^{(e)} \Delta}{3}$$

$$F j_i^{(e)} = - \oint_{(e)} \left( -\sigma_x \frac{\partial \tilde{u}^{(e)}}{\partial x} n_x^{(e)} - \sigma_y \frac{\partial \tilde{u}^{(e)}}{\partial y} n_y^{(e)} \right) dl$$

- Current density

$$j_x^{(e)} = -\sigma_x(x_c, y_c) \sum_{i=1}^3 \alpha_i \frac{\beta_i}{2\Delta} \quad \text{and} \quad j_y^{(e)} = -\sigma_y(x_c, y_c) \sum_{i=1}^3 \alpha_i \frac{\gamma_i}{2\Delta}$$



• **C<sup>0</sup>-quadratic isoparametric triangular element**

- Trial solution in the parent element,  $\phi_1(\xi, \eta) = \alpha_1 + \alpha_2\xi + \alpha_3\eta + \alpha_4\xi^2 + \alpha_5\xi\eta + \alpha_6\eta^2$

From  $\phi_1(\xi_1, \eta_1) = 1$  ,  $\phi_1(\xi_i, \eta_i) = 0$  for  $i = 2, \dots, 6$  and  $(\xi_1, \eta_1) = (0, 0)$  ,

$(\xi_2, \eta_2) = (1, 0)$  ,  $(\xi_3, \eta_3) = (0, 1)$  ,  $(\xi_4, \eta_4) = (0.5, 0)$  ,  $(\xi_5, \eta_5) = (0.5, 0.5)$  ,

$(\xi_6, \eta_6) = (0, 0.5)$ , we get  $\phi_1(\xi, \eta) = [1 - (\xi + \eta)][1 - 2(\xi + \eta)]$ .

Similarly,

$$\phi_2(\xi, \eta) = \xi(2\xi - 1)$$

$$\phi_3(\xi, \eta) = \eta(2\eta - 1),$$

$$\phi_4(\xi, \eta) = 4\xi[1 - (\xi + \eta)]$$

$$\phi_5(\xi, \eta) = 4\xi\eta$$

$$\phi_6(\xi, \eta) = 4\eta[1 - (\xi + \eta)]$$

- Define coordinate transformation as

$$x = \sum_{k=1}^6 x_k^{(e)} \phi_k(\xi) \quad \text{and} \quad y = \sum_{k=1}^6 y_k^{(e)} \phi_k(\xi)$$

Then,

$$J^{(e)}(\xi, \eta) = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} J_{11}^{(e)}(\xi, \eta) & J_{12}^{(e)}(\xi, \eta) \\ J_{21}^{(e)}(\xi, \eta) & J_{22}^{(e)}(\xi, \eta) \end{bmatrix} \quad \text{and} \quad |J^{(e)}(\xi, \eta)| = \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi} \frac{\partial x}{\partial \eta}$$

$$\begin{Bmatrix} \frac{\partial \phi_i}{\partial \xi} \\ \frac{\partial \phi_i}{\partial \eta} \end{Bmatrix} = [J^{(e)}] \begin{Bmatrix} \frac{\partial \phi_i}{\partial x} \\ \frac{\partial \phi_i}{\partial y} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \frac{\partial \phi_i}{\partial x} \\ \frac{\partial \phi_i}{\partial y} \end{Bmatrix} = [J^{(e)}]^{-1} \begin{Bmatrix} \frac{\partial \phi_i}{\partial \xi} \\ \frac{\partial \phi_i}{\partial \eta} \end{Bmatrix} \quad \text{with}$$

$$[J^{(e)}]^{-1} = \frac{1}{|J^{(e)}|} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{bmatrix}$$

Therefore,

$$\frac{\partial \phi_i}{\partial x} = \frac{1}{|J^{(e)}|} \left[ \frac{\partial y}{\partial \eta} \frac{\partial \phi_i}{\partial \xi} - \frac{\partial y}{\partial \xi} \frac{\partial \phi_i}{\partial \eta} \right] \quad \text{and} \quad \frac{\partial \phi_i}{\partial y} = \frac{1}{|J^{(e)}|} \left[ -\frac{\partial x}{\partial \eta} \frac{\partial \phi_i}{\partial \xi} + \frac{\partial x}{\partial \xi} \frac{\partial \phi_i}{\partial \eta} \right]$$

- Element matrix

$$K_{ij}^{(e)} = \int_0^1 \int_0^{1-\eta} \frac{\partial \phi_i^{(e)}}{\partial x} \sigma_x \frac{\partial \phi_j^{(e)}}{\partial x} |J^{(e)}| d\xi d\eta + \int_0^1 \int_0^{1-\eta} \frac{\partial \phi_i^{(e)}}{\partial y} \sigma_y \frac{\partial \phi_j^{(e)}}{\partial y} |J^{(e)}| d\xi d\eta$$

$$F_i^{(e)} = \int_0^1 \int_0^{1-\eta} f \phi_i^{(e)} |J^{(e)}| d\xi d\eta - \oint_{(e)} \left( -\sigma_x \frac{\partial \tilde{u}^{(e)}}{\partial x} n_x^{(e)} - \sigma_y \frac{\partial \tilde{u}^{(e)}}{\partial y} n_y^{(e)} \right) dl$$

- Numerical integration (see p. 596, Burnett)

$$K_{ij}^{(e)} = \frac{1}{2} \sum_{l=1}^n w_{nl} \left[ \left( \frac{\partial \phi_i^{(e)}}{\partial x} \sigma_x \frac{\partial \phi_j^{(e)}}{\partial x} + \frac{\partial \phi_i^{(e)}}{\partial y} \sigma_y \frac{\partial \phi_j^{(e)}}{\partial y} \right) |J^{(e)}| \right]_{(\xi_{nl}, \eta_{nl})}$$

$$Kf_i^{(e)} = \frac{1}{2} \sum_{l=1}^n w_{nl} \left[ f \phi_i^{(e)} |J^{(e)}| \right]_{(\xi_{nl}, \eta_{nl})}$$

### • $C^0$ -linear isoparametric quadrilateral element

- Interpolation functions on parent element are

$$\phi_1(\xi, \eta) = \frac{1}{4}(1-\xi)(1-\eta)$$

$$\phi_2(\xi, \eta) = \frac{1}{4}(1+\xi)(1-\eta)$$

$$\phi_3(\xi, \eta) = \frac{1}{4}(1+\xi)(1+\eta)$$

$$\phi_4(\xi, \eta) = \frac{1}{4}(1-\xi)(1+\eta)$$

with  $(\xi_1, \eta_1) = (-1, -1)$ ,  $(\xi_2, \eta_2) = (1, -1)$ ,  $(\xi_3, \eta_3) = (1, 1)$ ,  $(\xi_4, \eta_4) = (-1, 1)$ .

- Element matrix with numerical integration (see p.618, Burnett)

$$K_{ij}^{(e)} = \frac{1}{2} \sum_{k=1}^n \sum_{l=1}^n w_{nk} w_{nl} \left[ \left( \frac{\partial \phi_i^{(e)}}{\partial x} \sigma_x \frac{\partial \phi_j^{(e)}}{\partial x} + \frac{\partial \phi_i^{(e)}}{\partial y} \sigma_y \frac{\partial \phi_j^{(e)}}{\partial y} \right) \Big|_{J^{(e)}} \right]_{(\xi_{nl}, \eta_{nl})}$$

$$K f_i^{(e)} = \frac{1}{2} \sum_{k=1}^n \sum_{l=1}^n w_{nk} w_{nl} \left[ f \phi_i^{(e)} \Big|_{J^{(e)}} \right]_{(\xi_{nl}, \eta_{nl})}$$

• **C<sup>0</sup>-quadratic isoparametric quadrilateral element**

- Interpolation functions on parent element are

$$\phi_1(\xi, \eta) = \frac{1}{4}(1-\xi)(1-\eta)(-\xi-\eta-1)$$

$$\phi_2(\xi, \eta) = \frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta-1)$$

$$\phi_3(\xi, \eta) = \frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-1)$$

$$\phi_4(\xi, \eta) = \frac{1}{4}(1-\xi)(1+\eta)(-\xi+\eta-1)$$

$$\phi_5(\xi, \eta) = \frac{1}{2}(1-\xi^2)(1-\eta)$$

$$\phi_6(\xi, \eta) = \frac{1}{2}(1+\xi)(1-\eta^2)$$

$$\phi_7(\xi, \eta) = \frac{1}{2}(1-\xi^2)(1+\eta)$$

$$\phi_8(\xi, \eta) = \frac{1}{2}(1-\xi)(1-\eta^2)$$

with  $(\xi_1, \eta_1) = (-1, -1)$ ,  $(\xi_2, \eta_2) = (1, -1)$ ,  $(\xi_3, \eta_3) = (1, 1)$ ,  $(\xi_4, \eta_4) = (-1, 1)$ ,

$(\xi_5, \eta_5) = (0, -1)$ ,  $(\xi_6, \eta_6) = (1, 0)$ ,  $(\xi_7, \eta_7) = (0, 1)$ ,  $(\xi_8, \eta_8) = (-1, 0)$ .

- Element matrix with numerical integration (see p.618, Burnett)

$$K_{ij}^{(e)} = \frac{1}{2} \sum_{k=1}^n \sum_{l=1}^n w_{nk} w_{nl} \left[ \left( \frac{\partial \phi_i^{(e)}}{\partial x} \sigma_x \frac{\partial \phi_j^{(e)}}{\partial x} + \frac{\partial \phi_i^{(e)}}{\partial y} \sigma_y \frac{\partial \phi_j^{(e)}}{\partial y} \right) \Big|_{J^{(e)}} \right]_{(\xi_{nl}, \eta_{nk})}$$

$$Kf_i^{(e)} = \frac{1}{2} \sum_{k=1}^n \sum_{l=1}^n w_{nk} w_{nl} \left[ f \phi_i^{(e)} \Big|_{J^{(e)}} \right]_{(\xi_{nl}, \eta_{nk})}$$

### (D) Three-dimensional problem

- Governing equation (differential equation):

$$-\frac{\partial}{\partial x} \left( \sigma(x, y, z) \frac{\partial u(x, y, z)}{\partial x} \right) - \frac{\partial}{\partial y} \left( \sigma(x, y, z) \frac{\partial u(x, y, z)}{\partial y} \right) - \frac{\partial}{\partial z} \left( \sigma(x, y, z) \frac{\partial u(x, y, z)}{\partial z} \right) = f(x, y, z)$$

- Domain:  $s = (x, y, z) \in \Omega$

- Boundary:  $\partial\Omega$

- Outward unit normal vector:  $\{n\} = \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}$

- Essential boundary condition (Dirichlet boundary condition):  $u(s_a) = u_a, s_a \in \partial\Omega$

- Natural boundary condition (Neumann boundary condition):

$$-\sigma(x, y, z) \nabla u(x, y, z) \cdot \{n\} \Big|_{s_b} = j_b, s_b \in \partial\Omega$$

- Material property

(4) Homogeneous and isotropic on  $\Omega$ :  $\sigma(x, y, z) = \sigma$

(5) Homogeneous and isotropic within element:  $\sigma^{(e)}(x, y, z) = \sigma^{(e)}$

(6) Anisotropic:  $[\sigma] = \begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{bmatrix}$

#### • Element equation

- Following the same procedure as in one-dimensional problem with

$$\tilde{u}^{(e)}(x, y, z; a) = \sum_{j=1}^N a_j \phi_j^{(e)}(x, y, z),$$

$$[K]\{a\} = \{F\}$$

where

$$K_{ij}^{(e)} = \iiint_{(e)} \frac{\partial \phi_i^{(e)}}{\partial x} \sigma_x \frac{\partial \phi_j^{(e)}}{\partial x} dx dy dz + \iiint_{(e)} \frac{\partial \phi_i^{(e)}}{\partial y} \sigma_y \frac{\partial \phi_j^{(e)}}{\partial y} dx dy dz + \iiint_{(e)} \frac{\partial \phi_i^{(e)}}{\partial z} \sigma_z \frac{\partial \phi_j^{(e)}}{\partial z} dx dy dz$$

$$F_i^{(e)} = \iiint_{(e)} f \phi_i^{(e)} dx dy dz - \oint_{(e)} \left( -\sigma_x \frac{\partial \tilde{u}^{(e)}}{\partial x} n_x^{(e)} - \sigma_y \frac{\partial \tilde{u}^{(e)}}{\partial y} n_y^{(e)} - \sigma_z \frac{\partial \tilde{u}^{(e)}}{\partial z} n_z^{(e)} \right) dS$$

• **C<sup>0</sup>-linear tetrahedral element**

$$- \tilde{u}^{(e)}(x, y, z) = \alpha + \beta x + \gamma y + \delta z = \sum_{i=1}^4 a_i \phi_i^{(e)}(x, y, z) \quad \text{and} \quad \tilde{u}^{(e)}(x_i, y_i, z_i) = a_i \quad \text{with}$$

$$i = 1, \dots, 4$$

- Find  $\alpha, \beta, \gamma, \delta$  from

$$\begin{bmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

Then,  $\phi_i^{(e)}(x, y, z) = \frac{\alpha_i + \beta_i x + \gamma_i y + \delta_i z}{6V}$  with

$$\alpha_i = \begin{vmatrix} x_j & y_j & z_j \\ x_k & y_k & z_k \\ x_l & y_l & z_l \end{vmatrix}, \quad \beta_i = - \begin{vmatrix} 1 & y_j & z_j \\ 1 & y_k & z_k \\ 1 & y_l & z_l \end{vmatrix}, \quad \gamma_i = - \begin{vmatrix} x_j & 1 & z_j \\ x_k & 1 & z_k \\ x_l & 1 & z_l \end{vmatrix}, \quad \delta_i = - \begin{vmatrix} x_j & y_j & 1 \\ x_k & y_k & 1 \\ x_l & y_l & 1 \end{vmatrix},$$

$$6V = \begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix} = 6 \times \text{volume of element}$$

Apply cyclic indexing for  $j, k$ , and  $l$ .

- Element matrix:

$$K_{ij} = \frac{\sigma_x^{(e)}}{36V} \beta_i \beta_j + \frac{\sigma_y^{(e)}}{36V} \gamma_i \gamma_j + \frac{\sigma_z^{(e)}}{36V} \delta_i \delta_j$$

• **C<sup>0</sup> linear isoparametric triangular prism element**

- We find  $\phi_i(\xi, \eta, \zeta)$  from

$$\phi_i(\xi, \eta, \zeta) = \alpha_{i1} + \alpha_{i2}\xi + \alpha_{i3}\eta + \alpha_{i4}\zeta + \alpha_{i5}\xi\zeta + \alpha_{i6}\eta\zeta$$

for (0,0,-1), (1,0,-1), (0,1,-1), (0,0,1), (1,0,1), (0,1,1)

• **C<sup>0</sup> isoparametric hexahedral element**

- Linear element

$$\phi_i(\xi, \eta) = \frac{1}{8}(1 + \xi\xi_i)(1 + \eta\eta_i)(1 + \zeta\zeta_i)$$

with (-1,-1,-1), (1,-1,-1), (1,1,-1), (-1,1,-1), (-1,-1,1), (1,-1,1), (1,1,1), (-1,1,1)

- Quadratic element

$$\phi_i(\xi, \eta) = \frac{1}{8}(1 + \xi\xi_i)(1 + \eta\eta_i)(1 + \zeta\zeta_i)(\xi\xi_i + \eta\eta_i + \zeta\zeta_i - 2)$$

for (-1,-1,-1), (1,-1,-1), (1,1,-1), (-1,1,-1), (-1,-1,1), (1,-1,1), (1,1,1), (-1,1,1)

$$\phi_i(\xi, \eta) = \frac{1}{4}(1 - \xi^2)(1 + \eta\eta_i)(1 + \zeta\zeta_i)$$

for (0,-1,-1), (0,1,-1), (0,1,1), (0,-1,1)

$$\phi_i(\xi, \eta) = \frac{1}{4}(1 + \xi\xi_i)(1 - \eta^2)(1 + \zeta\zeta_i)$$

for (-1,0,-1), (1,0,-1), (1,0,1), (-1,0,1)

$$\phi_i(\xi, \eta) = \frac{1}{4}(1 + \xi\xi_i)(1 + \eta\eta_i)(1 - \zeta^2)$$

for (-1,-1,0), (1,-1,0), (1,1,0), (-1,1,0)