Finite Element Method (FEM)

(A) Electrostatic problem

- $\nabla\cdot\mathbf{D}=\rho$, $\mathbf{E}=\varepsilon\mathbf{D}=-\nabla u \;\Rightarrow\; \nabla\cdot\nabla u=\nabla^2 u=-\dfrac{\rho}{\varepsilon}$
- $\nabla\cdot\mathbf{J}=0$, $\mathbf{J}=\sigma\mathbf{E}=-\sigma\nabla u \;\Rightarrow\; \nabla\cdot(\sigma\nabla u)=0$

(B) One-dimensional problem

- Governing equation (differential equation): $-\dfrac{d}{dx}\left(\sigma(x)\dfrac{du(x)}{dx}\right)=f(x)$
- Domain: $x_a \le x \le x_b$
- Essential boundary condition (Dirichlet boundary condition): $u(x_a)=u_a$
- Natural boundary condition (Neumann boundary condition): $-\sigma(x)\dfrac{du(x)}{dx}\Big|_{x=x_b}=j_b$

• Trial solution

- $\tilde{u}(x;a)=\phi_0(x)+a_1\phi_1(x)+\cdots+a_N\phi_N(x)=\phi_0(x)+\sum_{j=1}^{N}a_j\phi_j(x)$
- a_i are values of $u(x)$ at some points
- We usually use polynomials for $\phi_i(x)$

• Residual for governing equation

- Residual: $R(x;a)=-\dfrac{d}{dx}\left(\sigma(x)\dfrac{d\tilde{u}(x)}{dx}\right)-f(x)$
- Orthogonality principle (Galerkin method):

$$\left\langle R(x;a),\phi_i(x)\right\rangle=\int_{x_a}^{x_b}R(x;a)\phi_i(x)dx=0\,,\quad i=1,\cdots,N$$

$$\Rightarrow\int_{x_a}^{x_b}\left\{-\frac{d}{dx}\left(\sigma(x)\frac{d\tilde{u}(x)}{dx}\right)-f(x)\right\}\phi_i(x)dx=0$$

$$\Rightarrow\int_{x_a}^{x_b}\frac{d}{dx}\left(\sigma(x)\frac{d\tilde{u}(x)}{dx}\right)\phi_i(x)dx=-\int_{x_a}^{x_b}f(x)\phi_i(x)dx$$

• Integration by part

$$\int_{x_a}^{x_b}\frac{d}{dx}(fg)dx=[gf]_{x_a}^{x_b}=\int_{x_a}^{x_b}\frac{df}{dx}gdx+\int_{x_a}^{x_b}\frac{dg}{dx}fdx \;\Rightarrow\; \int_{x_a}^{x_b}\frac{df}{dx}gdx=[gf]_{x_a}^{x_b}-\int_{x_a}^{x_b}\frac{dg}{dx}fdx$$

$$\int_{x_a}^{x_b}\frac{d}{dx}\left(\sigma(x)\frac{d\tilde{u}(x)}{dx}\right)\phi_i(x)dx=-\left[\left(-\sigma(x)\frac{d\tilde{u}(x)}{dx}\right)\phi_i(x)\right]_{x_a}^{x_b}-\int_{x_a}^{x_b}\sigma(x)\frac{d\tilde{u}(x)}{dx}\frac{d\phi_i(x)}{dx}dx$$

• System equations

$$\int_{x_a}^{x_b}\sigma(x)\frac{d\tilde{u}(x)}{dx}\frac{d\phi_i(x)}{dx}dx=-\left[\left(-\sigma(x)\frac{d\tilde{u}(x)}{dx}\right)\phi_i(x)\right]_{x_a}^{x_b}+\int_{x_a}^{x_b}f(x)\phi_i(x)dx$$

$$\frac{d\tilde{u}}{dx}=\frac{d\phi_0}{dx}+\sum_{j=1}^{N}a_j\frac{d\phi_j}{dx}$$

$$\begin{aligned}
&\int_{x_a}^{x_b}\sigma(x)\frac{d\tilde{u}(x)}{dx}\frac{d\phi_i(x)}{dx}dx\\
&=\int_{x_a}^{x_b}\sigma(x)\left\{\frac{d\phi_0}{dx}+\sum_{j=1}^{N}a_j\frac{d\phi_j}{dx}\right\}\frac{d\phi_i(x)}{dx}dx\\
&=\sum_{j=1}^{N}\left\{\int_{x_a}^{x_b}\sigma(x)\frac{d\phi_i(x)}{dx}\frac{d\phi_j(x)}{dx}dx\right\}a_j+\int_{x_a}^{x_b}\sigma(x)\frac{d\phi_i(x)}{dx}\frac{d\phi_0(x)}{dx}dx
\end{aligned}$$

$$\left.\begin{aligned}
&\sum_{j=1}^{N}\left\{\int_{x_a}^{x_b}\sigma(x)\frac{d\phi_i(x)}{dx}\frac{d\phi_j(x)}{dx}dx\right\}a_j\\
&=-\left[\left(-\sigma(x)\frac{d\tilde{u}(x)}{dx}\right)\phi_i(x)\right]_{x_a}^{x_b}+\int_{x_a}^{x_b}f(x)\phi_i(x)dx-\int_{x_a}^{x_b}\sigma(x)\frac{d\phi_i(x)}{dx}\frac{d\phi_0(x)}{dx}dx
\end{aligned}\right.\quad,\quad i=1,\cdots,N$$

$$\Downarrow$$

$$\left(\int_{x_a}^{x_b}\sigma(x)\frac{d\phi_1(x)}{dx}\frac{d\phi_1(x)}{dx}dx\right)a_1+\cdots+\left(\int_{x_a}^{x_b}\sigma(x)\frac{d\phi_1(x)}{dx}\frac{d\phi_N(x)}{dx}dx\right)a_N$$
$$=-\left[\left(-\sigma(x)\frac{d\tilde{u}(x)}{dx}\right)\phi_1(x)\right]_{x_a}^{x_b}+\int_{x_a}^{x_b}f(x)\phi_1(x)dx-\int_{x_a}^{x_b}\sigma(x)\frac{d\phi_1(x)}{dx}\frac{d\phi_0(x)}{dx}dx \qquad (i=1)$$

$$\vdots$$

$$\left(\int_{x_a}^{x_b}\sigma(x)\frac{d\phi_N(x)}{dx}\frac{d\phi_1(x)}{dx}dx\right)a_1+\cdots+\left(\int_{x_a}^{x_b}\sigma(x)\frac{d\phi_N(x)}{dx}\frac{d\phi_N(x)}{dx}dx\right)a_N$$
$$=-\left[\left(-\sigma(x)\frac{d\tilde{u}(x)}{dx}\right)\phi_N(x)\right]_{x_a}^{x_b}+\int_{x_a}^{x_b}f(x)\phi_N(x)dx-\int_{x_a}^{x_b}\sigma(x)\frac{d\phi_N(x)}{dx}\frac{d\phi_0(x)}{dx}dx \qquad (i=N)$$

$$\Downarrow$$

$$[K]\{a\}=\{F\}$$

$$K_{ij}=\int_{x_a}^{x_b}\sigma(x)\frac{d\phi_i(x)}{dx}\frac{d\phi_j(x)}{dx}dx \quad \text{and}$$

$$F_i=-\left[\left(-\sigma(x)\frac{d\tilde{u}(x)}{dx}\right)\phi_i(x)\right]_{x_a}^{x_b}+\int_{x_a}^{x_b}f(x)\phi_i(x)dx-\int_{x_a}^{x_b}\sigma(x)\frac{d\phi_i(x)}{dx}\frac{d\phi_0(x)}{dx}dx$$

• One linear element with two nodes

- Find $\phi_i(x)$
  (1) Let's assume $\tilde{u}(x;\alpha)=\alpha_1+\alpha_2 x$ on the domain $x_a\le x\le x_b$
  (2) At each node, $\tilde{u}(x_a;\alpha)=\alpha_1+\alpha_2 x_a=a_1$ and $\tilde{u}(x_b;\alpha)=\alpha_1+\alpha_2 x_b=a_2$

$$\Downarrow$$

$$\begin{bmatrix}1 & x_a\\ 1 & x_b\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}=\begin{bmatrix}a_1\\ a_2\end{bmatrix}$$

$$\Downarrow$$

$$\alpha_1 = \frac{x_b a_1 - x_a a_2}{x_b - x_a} \quad \text{and} \quad \alpha_2 = \frac{a_2 - a_1}{x_b - x_a}$$

$$\Downarrow$$

$$\tilde{u}(x;a) = \frac{x_b a_1 - x_a a_2}{x_b - x_a} + \frac{a_2 - a_1}{x_b - x_a} x = a_1 \frac{x_b - x}{x_b - x_a} + a_2 \frac{x - x_a}{x_b - x_a} = a_1\phi_1 + a_2\phi_2$$

$$\phi_1(x) = \frac{x_b - x}{x_b - x_a} \quad \text{and} \quad \phi_2(x) = \frac{x - x_a}{x_b - x_a} : \text{interpolation functions}$$

$$\phi_1(x_a) = 1,\ \phi_1(x_b) = 0 \ \text{ and } \ \phi_2(x_a) = 0,\ \phi_2(x_b) = 1 \ \Rightarrow\ \phi_j(x_i) = \delta_{ji}$$

(3) Other method for finding $\phi_i(x)$ from interpolation property

Let $\tilde{u}(x;a) = a_1\phi_1(x) + a_2\phi_2(x)$, $\phi_1(x) = \beta_1 + \beta_2 x$, and $\phi_1(x) = \gamma_1 + \gamma_2 x$

From $\phi_1(x_a) = 1$, $\phi_1(x_b) = 0$ and $\phi_2(x_a) = 0$, $\phi_2(x_b) = 1$, we get

$$\phi_1(x) = \frac{x_b - x}{x_b - x_a} \quad \text{and} \quad \phi_2(x) = \frac{x - x_a}{x_b - x_a}$$

- Evaluate system equations

(1) If f is not zero,

$$[K]\{a\} = \{F\}$$

$$\Downarrow$$

$$\begin{bmatrix} \frac{1}{2}\frac{x_b + x_a}{x_b - x_a} & -\frac{1}{2}\frac{x_b + x_a}{x_b - x_a} \\ -\frac{1}{2}\frac{x_b + x_a}{x_b - x_a} & \frac{1}{2}\frac{x_b + x_a}{x_b - x_a} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -\left[\left(-\sigma(x)\frac{d\tilde{u}(x)}{dx}\right)\phi_1(x)\right]_{x_a}^{x_b} + \int_{x_a}^{x_b} f(x)\phi_1(x)dx \\ -\left[\left(-\sigma(x)\frac{d\tilde{u}(x)}{dx}\right)\phi_2(x)\right]_{x_a}^{x_b} + \int_{x_a}^{x_b} f(x)\phi_2(x)dx \end{bmatrix}$$

(2) If f is zero,

$$[K]\{a\} = \{F\}$$

$$\Downarrow$$

$$\begin{bmatrix} \frac{1}{2}\frac{x_b+x_a}{x_b-x_a} & -\frac{1}{2}\frac{x_b+x_a}{x_b-x_a} \\ -\frac{1}{2}\frac{x_b+x_a}{x_b-x_a} & \frac{1}{2}\frac{x_b+x_a}{x_b-x_a} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -\left[\left(-\sigma(x)\frac{d\tilde{u}(x)}{dx}\right)\phi_1(x)\right]_{x_a}^{x_b} \\ -\left[\left(-\sigma(x)\frac{d\tilde{u}(x)}{dx}\right)\phi_2(x)\right]_{x_a}^{x_b} \end{bmatrix} = \begin{bmatrix} \left(-\sigma(x)\frac{d\tilde{u}(x)}{dx}\right)_{x_a} \\ -\left(-\sigma(x)\frac{d\tilde{u}(x)}{dx}\right)_{x_b} \end{bmatrix} = \begin{bmatrix} j_a \\ -j_b \end{bmatrix}$$

$$\text{with} \quad j_a + (-j_b) = 0$$

• Many linear elements (each element with two nodes)

- For k-th element, evaluate $\left[K^{(k)}\right]\left\{a^{(k)}\right\} = \left\{F^{(k)}\right\}$ with $k = 1, \cdots, E$

- Assemble system equations as

$$\begin{bmatrix}
K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 & 0 & \cdots & \cdots & 0 \\
K_{21}^{(1)} & K_{22}^{(1)}+K_{11}^{(2)} & K_{12}^{(2)} & 0 & 0 & \cdots & \cdots & 0 \\
0 & K_{21}^{(2)} & K_{22}^{(2)}+K_{11}^{(3)} & K_{12}^{(3)} & 0 & \cdots & \cdots & 0 \\
0 & 0 & K_{21}^{(3)} & K_{22}^{(3)}+K_{11}^{(4)} & K_{12}^{(4)} & \cdots & \cdots & 0 \\
0 & 0 & 0 & K_{21}^{(4)} & K_{22}^{(4)}+K_{11}^{(5)} & \cdots & \cdots & \vdots \\
0 & 0 & 0 & 0 & K_{21}^{(5)} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & K_{22}^{(E-1)}+K_{11}^{(E)} & K_{12}^{(E)} \\
0 & 0 & 0 & \cdots & \cdots & 0 & K_{21}^{(E)} & K_{22}^{(E)}
\end{bmatrix}
\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ \vdots \\ a_{E-1} \\ a_E \end{bmatrix}
=
\begin{bmatrix} F_1^{(1)} \\ F_2^{(1)}+F_1^{(2)} \\ F_2^{(2)}+F_1^{(3)} \\ F_2^{(3)}+F_1^{(4)} \\ F_2^{(5)}+F_1^{(6)} \\ \vdots \\ F_2^{(E-1)}+F_1^{(E)} \\ F_2^{(E)} \end{bmatrix}$$

using local node numbers, or

$$\begin{bmatrix}
K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 & 0 & \cdots & \cdots & 0 \\
K_{21}^{(1)} & K_{22}^{(1)}+K_{22}^{(2)} & K_{23}^{(2)} & 0 & 0 & \cdots & \cdots & 0 \\
0 & K_{32}^{(2)} & K_{33}^{(2)}+K_{33}^{(3)} & K_{34}^{(3)} & 0 & \cdots & \cdots & 0 \\
0 & 0 & K_{43}^{(3)} & K_{44}^{(3)}+K_{44}^{(4)} & K_{45}^{(4)} & \cdots & \cdots & 0 \\
0 & 0 & 0 & K_{54}^{(4)} & K_{55}^{(4)}+K_{55}^{(5)} & \cdots & \cdots & \vdots \\
0 & 0 & 0 & 0 & K_{65}^{(5)} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & K_{(E-1)(E-1)}^{(E-1)}+K_{(E-1)(E-1)}^{(E)} & K_{(E-1)E}^{(E)} \\
0 & 0 & 0 & \cdots & \cdots & 0 & K_{E(E-1)}^{(E)} & K_{EE}^{(E)}
\end{bmatrix}
\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ \vdots \\ a_{E-1} \\ a_E \end{bmatrix}
=
\begin{bmatrix} F_1^{(1)} \\ F_2^{(1)}+F_2^{(2)} \\ F_3^{(2)}+F_3^{(3)} \\ F_4^{(3)}+F_4^{(4)} \\ F_5^{(5)}+F_5^{(6)} \\ \vdots \\ F_{E-1}^{(E-1)}+F_{E-1}^{(E)} \\ F_2^{(E)} \end{bmatrix}$$

using global node numbers

• One-dimensional C^0-quadratic isoparametric element

- Define a master or parent element along $\xi -$axis with three nodes uniformed spaced at $\xi = -1, 0, +1$

- Define C^0-quadratic parent shape functions, $\phi_i(\xi)$ as

$$\phi_1(\xi) = \frac{(\xi-\xi_2)(\xi-\xi_3)}{(\xi_1-\xi_2)(\xi_1-\xi_3)} = \frac{\xi(\xi-1)}{2},$$

$$\phi_2(\xi) = \frac{(\xi-\xi_1)(\xi-\xi_3)}{(\xi_2-\xi_1)(\xi_2-\xi_3)} = -(\xi+1)(\xi-1),$$

$$\phi_3(\xi) = \frac{(\xi-\xi_1)(\xi-\xi_2)}{(\xi_3-\xi_1)(\xi_3-\xi_2)} = \frac{(\xi+1)\xi}{2}$$

- Define a coordinate transformation, $x = \chi^{(e)}(\xi)$ for each element (e) using the isoparametric approach as

$$x = \sum_{k=1}^{L} x_k^{(e)} \psi_k(\xi)$$

where $x_k^{(e)}$, $k = 1, \cdots, L$ are coordinates of L nodes in the element (e) and $\psi_k(\xi) = \phi_k(\xi)$ for $k = 1, \cdots, L$ (therefore, isoparametric). Therefore,

$$\begin{aligned} x &= \sum_{k=1}^{3} x_k^{(e)} \phi_k(\xi) \\ &= \frac{1}{2}\xi(\xi-1)x_1^{(e)} + (1+\xi)(1-\xi)x_2^{(e)} + \frac{1}{2}\xi(\xi+1)x_3^{(e)} \end{aligned}$$

$$(\xi_1 = -1 \Leftrightarrow x_1^{(e)}, \xi_2 = 0 \Leftrightarrow x_2^{(e)}, \xi_3 = 1 \Leftrightarrow x_3^{(e)})$$

- Evaluate element matrix, $[K^{(e)}]$ as

$$K_{ij}^{(e)} = \int_{x}^{x_3} \frac{d\phi_i^{(e)}(x)}{dx} \sigma(x) \frac{d\phi_j^{(e)}(x)}{dx} dx$$

where

$$\frac{d\phi_i^{(e)}(x)}{dx} = \frac{d\phi_i^{(e)}(x)}{d\xi}\frac{d\xi}{dx} = \frac{1}{J^{(e)}(\xi)}\frac{d\phi_i(\xi)}{d\xi} \quad \text{with} \quad J^{(e)}(\xi) = \frac{dx}{d\xi}$$

Note that $\phi_i^{(e)}(x) = \phi_i^{(e)}(\chi^{(e)}(\xi)) = \phi_i(\xi)$ and $\alpha(x) = \alpha(\chi^{(e)}(\xi)) = \alpha^{(e)}(\xi)$. Therefore,

$$K_{ij}^{(e)} = \int_{-1}^{+1} \left(\frac{1}{J^{(e)}(\xi)}\frac{d\phi_i(\xi)}{d\xi}\right) \sigma^{(e)}(\xi) \left(\frac{1}{J^{(e)}(\xi)}\frac{d\phi_j(\xi)}{d\xi}\right) J^{(e)}(\xi) d\xi$$

Similarly,

$$Ff_i^{(e)} = \int_{-1}^{+1} f^{(e)}(\xi)\phi_i(\xi)J^{(e)}(\xi)d\xi$$

- Numerical integration

$$\int_{-1}^{+1} I(\xi)d\xi \cong \sum_{l=1}^{N} w_{nl} I(\xi_{nl})$$ see Table 8.12 Gauss-Legendre integration (Burnett)

(C) Two-dimensional problem

- Governing equation (differential equation):

$$-\frac{\partial}{\partial x}\left(\sigma(x,y)\frac{\partial u(x,y)}{\partial x}\right) - \frac{\partial}{\partial y}\left(\sigma(x,y)\frac{\partial u(x,y)}{\partial y}\right) = f(x,y)$$

- Domain: $s = (x,y) \in \Omega$

- Boundary: $\partial\Omega$

- Outward unit normal vector: $\{n\} = \begin{Bmatrix} n_x \\ n_y \end{Bmatrix}$

- Essential boundary condition (Dirichlet boundary condition): $u(s_a) = u_a, s_a \in \partial\Omega$
- Natural boundary condition (Neumann boundary condition):

$$-\sigma(x,y)\nabla u(x,y)\cdot\{n\}\Big|_{s_b} = j_b, s_b \in \partial\Omega$$

- Material property
  (1) Homogeneous and isotropic on Ω: $\sigma(x,y) = \sigma$
  (2) Homogeneous and isotropic within element: $\sigma^{(e)}(x,y) = \sigma^{(e)}$
  (3) Anisotropic: $[\sigma] = \begin{bmatrix} \sigma_x & 0 \\ 0 & \sigma_y \end{bmatrix}$

• Element equation

- Following the same procedure as in one-dimensional problem with $\tilde{u}^{(e)}(x,y;a) = \sum_{j=1}^{N} a_j \phi_j^{(e)}(x,y)$,

$$[K]\{a\} = \{F\}$$

where

$$K_{ij}^{(e)} = \iint_{(e)} \frac{\partial \phi_i^{(e)}}{\partial x} \sigma_x \frac{\partial \phi_j^{(e)}}{\partial x} dxdy + \iint_{(e)} \frac{\partial \phi_i^{(e)}}{\partial y} \sigma_y \frac{\partial \phi_j^{(e)}}{\partial y} dxdy$$

$$F_i^{(e)} = \iint_{(e)} f \phi_i^{(e)} dxdy - \oint_{(e)} \left(-\sigma_x \frac{\partial \tilde{u}^{(e)}}{\partial x} n_x^{(e)} - \sigma_y \frac{\partial \tilde{u}^{(e)}}{\partial y} n_y^{(e)} \right) dl$$

• **C^0-linear triangular element**

- $\tilde{u}^{(e)}(x,y) = \alpha + \beta x + \gamma y = \sum_{i=1}^{3} a_i \phi_i^{(e)}(x,y)$ and $\tilde{u}^{(e)}(x_i, y_i) = a_i$ with $i = 1, \cdots, 3$
- Find α, β, γ from

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Then, $\phi_i^{(e)}(x,y) = \dfrac{\alpha_i + \beta_i x + \gamma_i y}{2\Delta}$ with

$$\alpha_i = x_j y_k - x_k y_j, \quad \beta_i = y_k - y_j, \quad \gamma_i = x_k - x_j, \quad \Delta = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \text{area of element}$$

- Element matrix

$$K_{ij} = \frac{\sigma_x^{(e)}}{4\Delta} \beta_i \beta_j + \frac{\sigma_y^{(e)}}{4\Delta} \gamma_i \gamma_j$$

$$Ff_i^{(e)} = \frac{f^{(e)} \Delta}{3}$$

$$Fj_i^{(e)} = -\oint_{(e)} \left(-\sigma_x \frac{\partial \tilde{u}^{(e)}}{\partial x} n_x^{(e)} - \sigma_y \frac{\partial \tilde{u}^{(e)}}{\partial y} n_y^{(e)} \right) dl$$

- Current density

$$j_x^{(e)} = -\sigma_x(x_c, y_c) \sum_{i=1}^{3} \alpha_i \frac{\beta_i}{2\Delta} \quad \text{and} \quad j_y^{(e)} = -\sigma_y(x_c, y_c) \sum_{i=1}^{3} \alpha_i \frac{\gamma_i}{2\Delta}$$

• C^0-quadratic isoparametric triangular element

- Trial solution in the parent element, $\phi_1(\xi,\eta)=\alpha_1+\alpha_2\xi+\alpha_3\eta+\alpha_4\xi^2+\alpha_5\xi\eta+\alpha_6\eta^2$

From $\phi_1(\xi_1,\eta_1)=1$, $\phi_i(\xi_i,\eta_i)=0$ for $i=2,\cdots,6$ and $(\xi_1,\eta_1)=(0,0)$, $(\xi_2,\eta_2)=(1,0)$, $(\xi_3,\eta_3)=(0,1)$, $(\xi_4,\eta_4)=(0.5,0)$, $(\xi_5,\eta_5)=(0.5,0.5)$, $(\xi_6,\eta_6)=(0,0.5)$, we get $\phi_1(\xi,\eta)=\left[1-(\xi+\eta)\right]\left[1-2(\xi+\eta)\right]$.

Similarly,

$$\phi_2(\xi,\eta)=\xi(2\xi-1)$$

$$\phi_3(\xi,\eta)=\eta(2\eta-1),$$

$$\phi_4(\xi,\eta)=4\xi\left[1-(\xi+\eta)\right]$$

$$\phi_5(\xi,\eta)=4\xi\eta$$

$$\phi_6(\xi,\eta)=4\eta\left[1-(\xi+\eta)\right]$$

- Define coordinate transformation as

$$x=\sum_{k=1}^{6}x_k^{(e)}\phi_k(\xi) \quad \text{and} \quad y=\sum_{k=1}^{6}y_k^{(e)}\phi_k(\xi)$$

Then,

$$J^{(e)}(\xi,\eta)=\begin{bmatrix}\dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi}\\[2mm] \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta}\end{bmatrix}=\begin{bmatrix}J_{11}^{(e)}(\xi,\eta) & J_{12}^{(e)}(\xi,\eta)\\ J_{21}^{(e)}(\xi,\eta) & J_{22}^{(e)}(\xi,\eta)\end{bmatrix} \text{ and } \left|J^{(e)}(\xi,\eta)\right|=\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta}-\frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta}$$

$$\begin{Bmatrix}\dfrac{\partial \phi_i}{\partial \xi}\\[2mm] \dfrac{\partial \phi_i}{\partial \eta}\end{Bmatrix}=\left[J^{(e)}\right]\begin{Bmatrix}\dfrac{\partial \phi_i}{\partial x}\\[2mm] \dfrac{\partial \phi_i}{\partial y}\end{Bmatrix} \text{ and } \begin{Bmatrix}\dfrac{\partial \phi_i}{\partial x}\\[2mm] \dfrac{\partial \phi_i}{\partial y}\end{Bmatrix}=\left[J^{(e)}\right]^{-1}\begin{Bmatrix}\dfrac{\partial \phi_i}{\partial \xi}\\[2mm] \dfrac{\partial \phi_i}{\partial \eta}\end{Bmatrix} \text{ with}$$

$$\left[J^{(e)}\right]^{-1} = \frac{1}{\left|J^{(e)}\right|}\begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial y}{\partial \xi} \\ -\dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \xi} \end{bmatrix}$$

Therefore,

$$\frac{\partial \phi_i}{\partial x} = \frac{1}{\left|J^{(e)}\right|}\left[\frac{\partial y}{\partial \eta}\frac{\partial \phi_i}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial \phi_i}{\partial \eta}\right] \text{ and } \frac{\partial \phi_i}{\partial y} = \frac{1}{\left|J^{(e)}\right|}\left[-\frac{\partial x}{\partial \eta}\frac{\partial \phi_i}{\partial \xi} + \frac{\partial x}{\partial \xi}\frac{\partial \phi_i}{\partial \eta}\right]$$

- Element matrix

$$K_{ij}^{(e)} = \int_0^1\int_0^{1-\eta}\frac{\partial \phi_i^{(e)}}{\partial x}\sigma_x\frac{\partial \phi_j^{(e)}}{\partial x}\left|J^{(e)}\right| d\xi d\eta + \int_0^1\int_0^{1-\eta}\frac{\partial \phi_i^{(e)}}{\partial y}\sigma_y\frac{\partial \phi_j^{(e)}}{\partial y}\left|J^{(e)}\right| d\xi d\eta$$

$$F_i^{(e)} = \int_0^1\int_0^{1-\eta} f\phi_i^{(e)}\left|J^{(e)}\right| d\xi d\eta - \oint_{(e)}\left(-\sigma_x\frac{\partial \tilde{u}^{(e)}}{\partial x}n_x^{(e)} - \sigma_y\frac{\partial \tilde{u}^{(e)}}{\partial y}n_y^{(e)}\right)dl$$

- Numerical integration (see p. 596, Burnett)

$$K_{ij}^{(e)} = \frac{1}{2}\sum_{l=1}^{n} w_{nl}\left[\left(\frac{\partial \phi_i^{(e)}}{\partial x}\sigma_x\frac{\partial \phi_j^{(e)}}{\partial x} + \frac{\partial \phi_i^{(e)}}{\partial y}\sigma_y\frac{\partial \phi_j^{(e)}}{\partial y}\right)\left|J^{(e)}\right|\right]_{(\xi_{nl},\eta_{nl})}$$

$$Kf_i^{(e)} = \frac{1}{2}\sum_{l=1}^{n} w_{nl}\left[f\phi_i^{(e)}\left|J^{(e)}\right|\right]_{(\xi_{nl},\eta_{nl})}$$

• C^0-linear isoparametric quadrilateral element

- Interpolation functions on parent element are

$$\phi_1(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta)$$

$$\phi_2(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta)$$

$$\phi_3(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta)$$

$$\phi_4(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta)$$

with $(\xi_1,\eta_1)=(-1,-1)$, $(\xi_2,\eta_2)=(1,-1)$, $(\xi_3,\eta_3)=(1,1)$, $(\xi_4,\eta_4)=(-1,1)$.

- Element matrix with numerical integration (see p.618, Burnett)

$$K_{ij}^{(e)}=\frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n}w_{nk}w_{nl}\left[\left(\frac{\partial\phi_i^{(e)}}{\partial x}\sigma_x\frac{\partial\phi_j^{(e)}}{\partial x}+\frac{\partial\phi_i^{(e)}}{\partial y}\sigma_y\frac{\partial\phi_j^{(e)}}{\partial y}\right)\left|J^{(e)}\right|\right]_{(\xi_{nl},\eta_{nk})}$$

$$Kf_i^{(e)}=\frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n}w_{nk}w_{nl}\left[f\phi_i^{(e)}\left|J^{(e)}\right|\right]_{(\xi_{nl},\eta_{nk})}$$

• **C^0-quadratic isoparametric quadrilateral element**

- Interpolation functions on parent element are

$$\phi_1(\xi,\eta)=\frac{1}{4}(1-\xi)(1-\eta)(-\xi-\eta-1)$$

$$\phi_2(\xi,\eta)=\frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta-1)$$

$$\phi_3(\xi,\eta)=\frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-1)$$

$$\phi_4(\xi,\eta)=\frac{1}{4}(1-\xi)(1+\eta)(-\xi+\eta-1)$$

$$\phi_5(\xi,\eta)=\frac{1}{2}(1-\xi^2)(1-\eta)$$

$$\phi_6(\xi,\eta)=\frac{1}{2}(1+\xi)(1-\eta^2)$$

$$\phi_7(\xi,\eta)=\frac{1}{2}(1-\xi^2)(1+\eta)$$

$$\phi_8(\xi,\eta)=\frac{1}{2}(1-\xi)(1-\eta^2)$$

with $(\xi_1,\eta_1)=(-1,-1)$, $(\xi_2,\eta_2)=(1,-1)$, $(\xi_3,\eta_3)=(1,1)$, $(\xi_4,\eta_4)=(-1,1)$, $(\xi_5,\eta_5)=(0,-1)$, $(\xi_6,\eta_6)=(1,0)$, $(\xi_7,\eta_7)=(0,1)$, $(\xi_8,\eta_8)=(-1,0)$.

- Element matrix with numerical integration (see p.618, Burnett)

$$K_{ij}^{(e)} = \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n} w_{nk} w_{nl} \left[\left(\frac{\partial \phi_i^{(e)}}{\partial x} \sigma_x \frac{\partial \phi_j^{(e)}}{\partial x} + \frac{\partial \phi_i^{(e)}}{\partial y} \sigma_y \frac{\partial \phi_j^{(e)}}{\partial y} \right) \left| J^{(e)} \right| \right]_{(\xi_{nl},\eta_{nk})}$$

$$Kf_i^{(e)} = \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n} w_{nk} w_{nl} \left[f \phi_i^{(e)} \left| J^{(e)} \right| \right]_{(\xi_{nl},\eta_{nk})}$$

(D) Three-dimensional problem

- Governing equation (differential equation):

$$-\frac{\partial}{\partial x}\left(\sigma(x,y,z) \frac{\partial u(x,y,z)}{\partial x} \right) - \frac{\partial}{\partial y}\left(\sigma(x,y,z) \frac{\partial u(x,y,z)}{\partial y} \right) - \frac{\partial}{\partial z}\left(\sigma(x,y,z) \frac{\partial u(x,y,z)}{\partial z} \right) = f(x,y,z)$$

- Domain: $s = (x,y,z) \in \Omega$

- Boundary: $\partial\Omega$

- Outward unit normal vector: $\{n\} = \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}$

- Essential boundary condition (Dirichlet boundary condition): $u(s_a) = u_a, s_a \in \partial\Omega$
- Natural boundary condition (Neumann boundary condition):

$$-\sigma(x,y,z)\nabla u(x,y,z) \cdot \{n\}\big|_{s_b} = j_b, s_b \in \partial\Omega$$

- Material property
  (4) Homogeneous and isotropic on Ω: $\sigma(x,y,z) = \sigma$
  (5) Homogeneous and isotropic within element: $\sigma^{(e)}(x,y,z) = \sigma^{(e)}$

  (6) Anisotropic: $[\sigma] = \begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{bmatrix}$

• Element equation

- Following the same procedure as in one-dimensional problem with $\tilde{u}^{(e)}(x,y,z;a) = \sum_{j=1}^{N} a_j \phi_j^{(e)}(x,y,z)$,

$$[K]\{a\}=\{F\}$$

where

$$K_{ij}^{(e)}=\iiint_{(e)}\frac{\partial\phi_i^{(e)}}{\partial x}\sigma_x\frac{\partial\phi_j^{(e)}}{\partial x}dxdydz+\iiint_{(e)}\frac{\partial\phi_i^{(e)}}{\partial y}\sigma_y\frac{\partial\phi_j^{(e)}}{\partial y}dxdydz+\iiint_{(e)}\frac{\partial\phi_i^{(e)}}{\partial z}\sigma_z\frac{\partial\phi_j^{(e)}}{\partial z}dxdydz$$

$$F_i^{(e)}=\iiint_{(e)}f\phi_i^{(e)}dxdydz-\oiint_{(e)}\left(-\sigma_x\frac{\partial\tilde{u}^{(e)}}{\partial x}n_x^{(e)}-\sigma_y\frac{\partial\tilde{u}^{(e)}}{\partial y}n_y^{(e)}-\sigma_z\frac{\partial\tilde{u}^{(e)}}{\partial z}n_z^{(e)}\right)dS$$

• C^0-linear tetrahedral element

- $\tilde{u}^{(e)}(x,y,z)=\alpha+\beta x+\gamma y+\delta z=\sum_{i=1}^{4}a_i\phi_i^{(e)}(x,y,z)$ and $\tilde{u}^{(e)}(x_i,y_i,z_i)=a_i$ with $i=1,\cdots,4$
- Find $\alpha,\beta,\gamma,\delta$ from

$$\begin{bmatrix}1 & x_1 & y_1 & z_1\\1 & x_2 & y_2 & z_2\\1 & x_3 & y_3 & z_3\\1 & x_4 & y_4 & z_4\end{bmatrix}\begin{bmatrix}\alpha\\\beta\\\gamma\\\delta\end{bmatrix}=\begin{bmatrix}a_1\\a_2\\a_3\\a_4\end{bmatrix}$$

Then, $\phi_i^{(e)}(x,y,z)=\dfrac{\alpha_i+\beta_i x+\gamma_i y+\delta_i z}{6V}$ with

$$\alpha_i=\begin{vmatrix}x_j & y_j & z_j\\x_k & y_k & z_k\\x_l & y_l & z_l\end{vmatrix},\quad \beta_i=-\begin{vmatrix}1 & y_j & z_j\\1 & y_k & z_k\\1 & y_l & z_l\end{vmatrix},\quad \gamma_i=-\begin{vmatrix}x_j & 1 & z_j\\x_k & 1 & z_k\\x_l & 1 & z_l\end{vmatrix},\quad \delta_i=-\begin{vmatrix}x_j & y_j & 1\\x_k & y_k & 1\\x_l & y_l & 1\end{vmatrix},$$

$$6V=\begin{vmatrix}1 & x_1 & y_1 & z_1\\1 & x_2 & y_2 & z_2\\1 & x_3 & y_3 & z_3\\1 & x_4 & y_4 & z_4\end{vmatrix}=6\times\text{volume of element}$$

Apply cyclic indexing for j, k, and l.

- Element matrix:

$$K_{ij} = \frac{\sigma_x^{(e)}}{36V}\beta_i\beta_j + \frac{\sigma_y^{(e)}}{36V}\gamma_i\gamma_j + \frac{\sigma_z^{(e)}}{36V}\delta_i\delta_j$$

• C^0 linear isoparametric triangular prism element

- We find $\phi_i(\xi,\eta,\zeta)$ from

$$\phi_i(\xi,\eta,\zeta) = \alpha_{i1} + \alpha_{i2}\xi + \alpha_{i3}\eta + \alpha_{i4}\zeta + \alpha_{i5}\xi\zeta + \alpha_{i6}\eta\zeta$$

for (0,0,-1), (1,0,-1), (0,1,-1), (0,0,1), (1,0,1), (0,1,1)

• C^0 isoparametric hexahedral element

- Linear element

$$\phi_i(\xi,\eta) = \frac{1}{8}(1+\xi\xi_i)(1+\eta\eta_i)(1+\zeta\zeta_i)$$

with (-1,-1,-1), (1,-1,-1), (1,1,-1), (-1,1,-1), (-1,-1,1), (1,-1,1), (1,1,1), (-1,1,1)

- Quadratic element

$$\phi_i(\xi,\eta) = \frac{1}{8}(1+\xi\xi_i)(1+\eta\eta_i)(1+\zeta\zeta_i)(\xi\xi_i + \eta\eta_i + \zeta\zeta_i - 2)$$

for (-1,-1,-1), (1,-1,-1), (1,1,-1), (-1,1,-1), (-1,-1,1), (1,-1,1), (1,1,1), (-1,1,1)

$$\phi_i(\xi,\eta) = \frac{1}{4}(1-\xi^2)(1+\eta\eta_i)(1+\zeta\zeta_i)$$

for (0,-1,-1), (0,1,-1), (0,1,1), (0,-1,1)

$$\phi_i(\xi,\eta) = \frac{1}{4}(1+\xi\xi_i)(1-\eta^2)(1+\zeta\zeta_i)$$

for (-1,0,-1), (1,0,-1), (1,0,1), (-1,0,1)

$$\phi_i(\xi,\eta) = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)(1-\zeta^2)$$

for (-1,-1,0), (1,-1,0), (1,1,0), (-1,1,0)