

Plonsey & Barr Bioelectricity 3rd ed.

2. Vector Analysis

$\left\{ \begin{array}{l} \text{scalar} : \text{only magnitude} \\ \text{vector} : \text{magnitude \& direction} \end{array} \right.$

$\left\{ \begin{array}{l} \text{scalar field} : \text{scalar function of position} \\ \text{vector field} : \text{vector " " " "} \end{array} \right.$

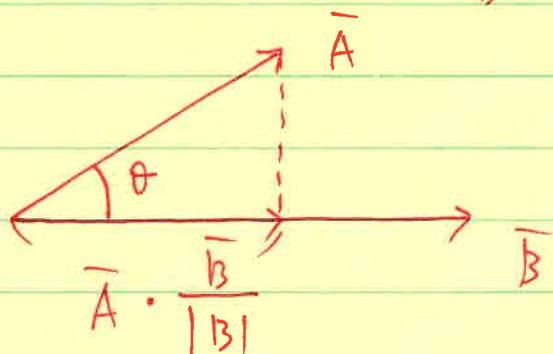
Sum : $\vec{C} = \vec{A} + \vec{B}$

~~Scalar product~~ & scaling : $\vec{B} = m \vec{A}$

Unit vector : $\hat{a} = \frac{\vec{A}}{|\vec{A}|}$

Dot product (or Scalar product)

$$\begin{aligned} \vec{A} \cdot \vec{B} &= |\vec{A}| |\vec{B}| \cos \theta = AB \cos \theta \\ &= \vec{B} \cdot \vec{A} = A_x B_x + A_y B_y + A_z B_z \end{aligned}$$



$A \cdot \frac{B}{|B|}$ component of \vec{A} along \vec{B}

or

projection of \vec{A} along \vec{B}

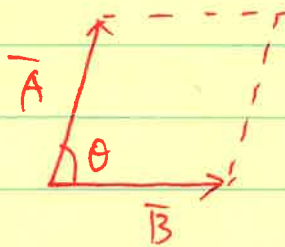
(Note : $\theta = 90^\circ$ and $\theta = 0^\circ$)

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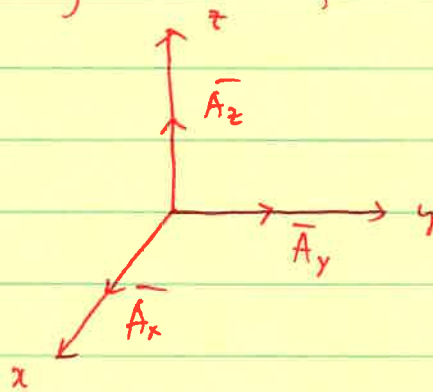
Cross product (or Vector product)

$$\vec{C} = \vec{A} \times \vec{B}$$

$$|\vec{C}| = |\vec{A}| |\vec{B}| \sin \theta \quad : \text{ area of parallelogram}$$



Direction of \vec{C} follows RHR rule $\left(\begin{array}{c} \vec{C} \\ \otimes \end{array} \right)$ "into the page"



$$\vec{A}_x = \vec{A}_y \times \vec{A}_z$$

$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y) \vec{a}_x + (A_z B_x - A_x B_z) \vec{a}_y + (A_x B_y - A_y B_x) \vec{a}_z$$

$$= \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Components of Vector : $\vec{A} = A_x \vec{a}_x + A_y \vec{a}_y + A_z \vec{a}_z$



③

* Gradient

$\Phi(x, y, z)$: scalar field

- single-valued
- continuous
- differentiable

$\Phi(x, y, z) = C$ defines a surface
(equipotential surface)
cf. contour map

$\vec{dl} \rightarrow P_2(x+dx, y+dy, z+dz)$
 $P_1(x, y, z)$

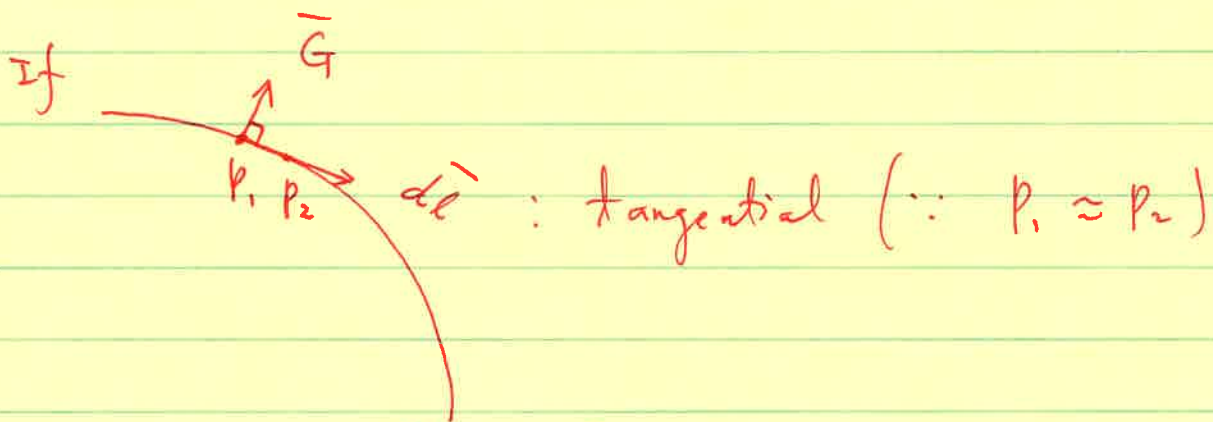
$$\vec{dl} = dx \vec{a}_x + dy \vec{a}_y + dz \vec{a}_z$$

$$\begin{aligned} d\Phi &= \Phi(x+dx, y+dy, z+dz) - \Phi(x, y, z) \\ &= \frac{\partial \Phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy + \frac{\partial \Phi}{\partial z} dz \\ &= \Phi(P_2) - \Phi(P_1) = \text{total derivative at } P_1 \end{aligned}$$

$$\vec{G} \triangleq \frac{\partial \Phi}{\partial x} \vec{a}_x + \frac{\partial \Phi}{\partial y} \vec{a}_y + \frac{\partial \Phi}{\partial z} \vec{a}_z : \text{Gradient of } \Phi$$

$$d\Phi = \vec{G} \cdot \vec{dl}$$

(4)

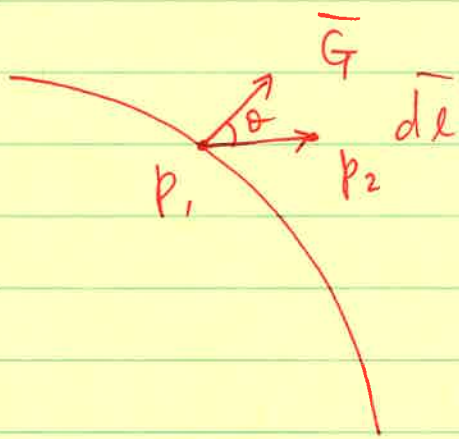


$$\Phi(p_1) = \Phi(p_2) = C_1$$

$$d\Phi = 0 = \vec{G} \cdot d\vec{l} \Rightarrow \vec{G} \perp d\vec{l}$$

$\Rightarrow \vec{G}$ is normal to surface C_1 at p_1

In general,



$$d\Phi = \vec{G} \cdot d\vec{l}$$

$$= |\vec{G}| |d\vec{l}| \cos\theta$$

$$\Rightarrow \boxed{\frac{d\Phi}{dl} = G \cos\theta}$$

$$\Phi(x, y, z) = C_1$$

Derivative of G in the direction l

(Directional derivative)

When $\theta = 0$, $d\vec{l} = \vec{n}$: outward unit normal vector

$$G = \frac{d\Phi}{dn}$$

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$$\bar{G} = G \cdot \bar{n} \quad : \quad \text{Gradient}$$

Direction ^(\bar{n}) is normal to the iso-surface
G (magnitude) is the maximum rate of increase in Φ

cf, Steepest descent method

* Del operator

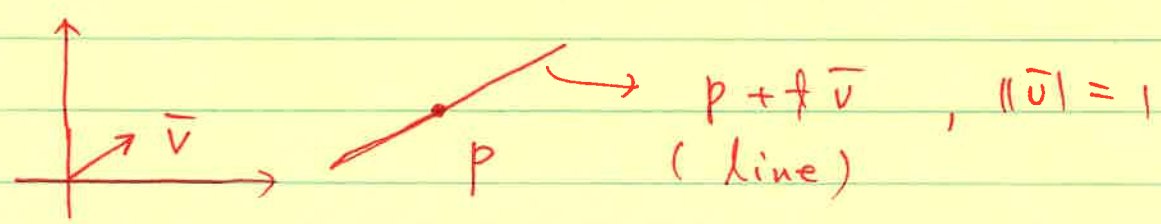
$$\nabla \equiv \frac{\partial}{\partial x} \bar{a}_x + \frac{\partial}{\partial y} \bar{a}_y + \frac{\partial}{\partial z} \bar{a}_z$$

$$\bar{G} = \nabla \Phi = \frac{\partial \Phi}{\partial x} \bar{a}_x + \frac{\partial \Phi}{\partial y} \bar{a}_y + \frac{\partial \Phi}{\partial z} \bar{a}_z$$

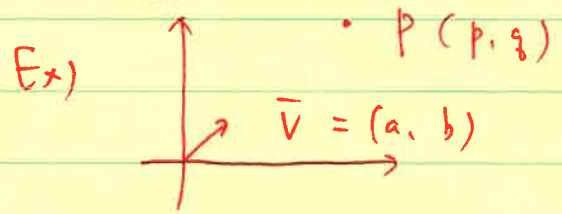
$$|\bar{G}| = G = |\nabla \Phi| = \sqrt{\left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2}$$

$\left\{ \begin{array}{l} \Phi : \text{ scalar, potential, effort variable} \\ \nabla \Phi : \text{ vector, flow, flow variable} \end{array} \right.$

* Directional Derivative and Gradient
f: differentiable, P: point



$$\frac{d(p + t\bar{v})}{dt} = \bar{v}$$



$$p + t\bar{v} = (p + ta, q + tb)$$

$$\frac{d(p + t\bar{v})}{dt} = (a, b) = \bar{v}$$

Let $g(t) = f(p + t\bar{v})$: change of f along the line $p + t\bar{v}$

$$\frac{df(p + t\bar{v})}{dt} = \nabla f(p + t\bar{v}) \cdot \frac{d(p + t\bar{v})}{dt} = \nabla f(p + t\bar{v}) \cdot \bar{v}$$

(chain rule)

If $t=0$, $\nabla f(p) \cdot \bar{v} = D_{\bar{v}} f(p)$
 \Rightarrow Directional derivative of f in the direction of \bar{v} at P

If $\nabla f(p) \parallel \bar{v}$, $|D_{\bar{v}} f(p)|$ is maximal.

\Rightarrow Direction of $\nabla f(p)$ is the steepest direction of f at P .

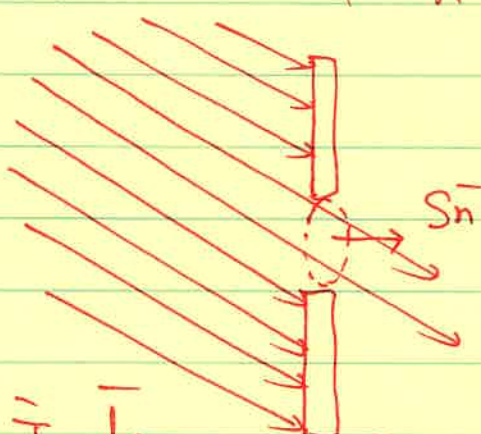
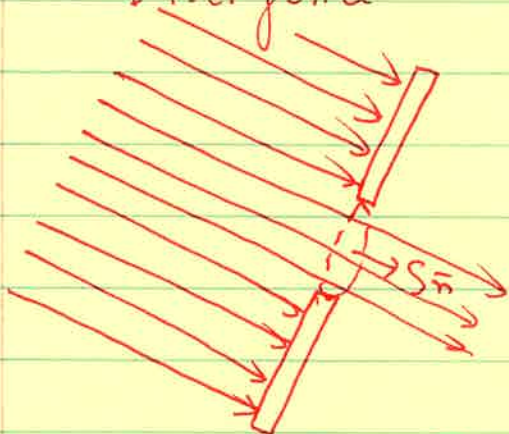
①

vector field

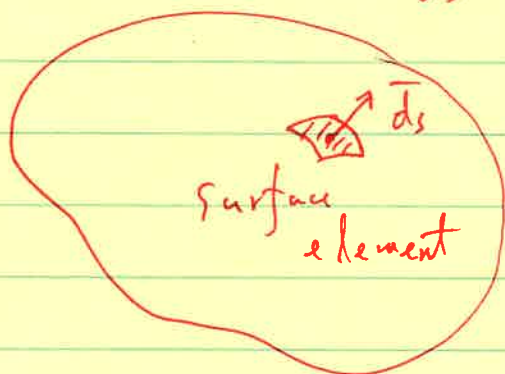
* Divergence

Assume

$$\vec{J}(x, y, z)$$

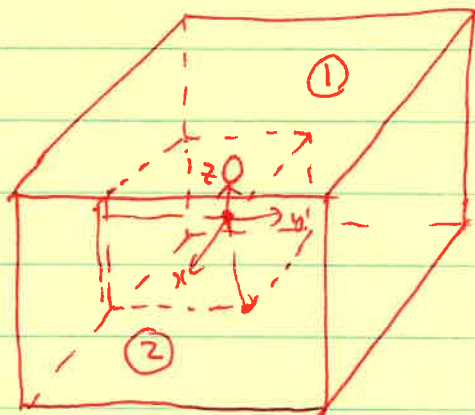


$$I = \oint_S \vec{J} \cdot \vec{ds} = \vec{J} \cdot S\vec{n} = JS \cos \theta$$



$$\vec{ds} = ds \cdot \vec{n}$$

\vec{n} : outward unit normal vector



$$O(x, y, z)$$

$$\text{Outflow}|_{\text{②}} = \left[J_x + \frac{1}{2} \frac{\partial J_x}{\partial x} dx \right] dy dz$$

(6)

$$\text{Outflow}|_{(1)} = \left[J_x - \frac{1}{2} \frac{\partial J_x}{\partial x} dx \right] (-dydz)$$

$$\oint_S \vec{J} \cdot \vec{ds} = \left[\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} \right] dx dy dz$$

$$\begin{aligned} \text{div } \vec{J} &\equiv \lim_{V \rightarrow 0} \frac{\oint \vec{J} \cdot \vec{ds}}{V} \\ &= \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} \\ &= \nabla \cdot \vec{J} \end{aligned}$$

$$\left\{ \begin{array}{l} \text{div } \vec{J} = 0 \quad : \quad \text{source (or sink) free} \\ \quad \quad \quad \quad - \text{continuity of current} \\ \quad \quad \quad \quad - \text{conservation of charge} \\ \text{div } \vec{J} > 0 \quad : \quad \text{source} \\ \text{div } \vec{J} < 0 \quad : \quad \text{sink} \end{array} \right.$$

* Laplacian

Φ : potential

$\nabla \cdot \Phi$: flow

$$\nabla \cdot \nabla \Phi = \nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

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In a source-free region,

$$\nabla^2 \Phi = 0 \quad \text{: Laplace equation}$$

of. Measure Φ and compute $\nabla^2 \Phi$.

If $\nabla^2 \Phi > 0$ at P , source is there.

If $\nabla^2 \Phi < 0$ at θ , sink is there.

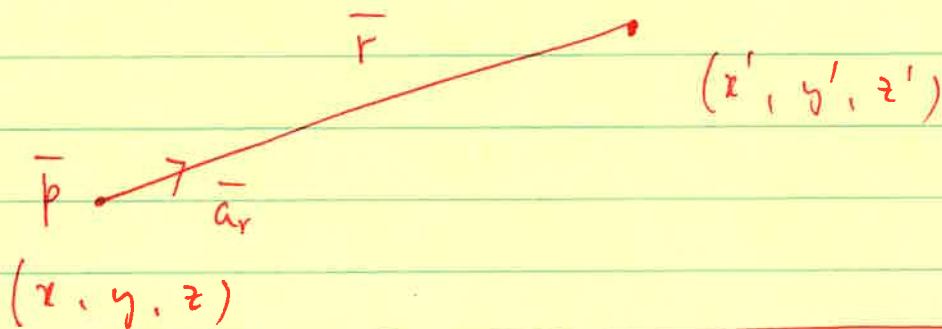
* Note

$$\nabla \cdot (\Phi \bar{A}) = \bar{A} \cdot \nabla \Phi + \Phi \nabla \cdot \bar{A}$$

$$\nabla (\Phi \psi) = \Phi \nabla \psi + \psi \nabla \Phi$$

$$\nabla^2 r = 0 \quad \text{where } r = \sqrt{x^2 + y^2 + z^2}$$

* Source and field points



$$r = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$
$$= r(x, y, z, x', y', z')$$

$$\bar{a}_r = \frac{(x'-x)\bar{a}_x + (y'-y)\bar{a}_y + (z'-z)\bar{a}_z}{r}$$

(10)

$$\nabla\left(\frac{1}{r}\right) = \frac{\partial}{\partial x}\left(\frac{1}{r}\right)\bar{a}_x + \frac{\partial}{\partial y}\left(\frac{1}{r}\right)\bar{a}_y + \frac{\partial}{\partial z}\left(\frac{1}{r}\right)\bar{a}_z$$

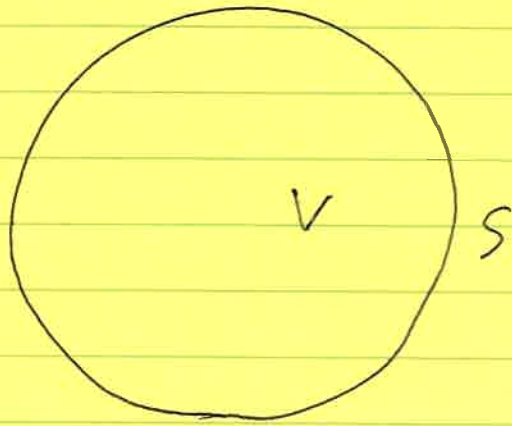
$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) &= \frac{\partial}{\partial x} \left\{ (x-x')^2 + (y-y')^2 + (z-z')^2 \right\}^{-\frac{1}{2}} \\ &= -\frac{1}{2} \left\{ (x-x')^2 + (y-y')^2 + (z-z')^2 \right\}^{-\frac{3}{2}} \times 2(x-x') \\ &= -\frac{x-x'}{r^3}\end{aligned}$$

$$\begin{aligned}\nabla\left(\frac{1}{r}\right) &= -\frac{(x-x')\bar{a}_x + (y-y')\bar{a}_y + (z-z')\bar{a}_z}{r^3} \\ &= \left(-\frac{1}{r^2}\right) \times (-\bar{a}_r) \\ &= \frac{\bar{a}_r}{r^2}\end{aligned}$$

$$\nabla'\left(\frac{1}{r}\right) = -\nabla\left(\frac{1}{r}\right) = -\frac{\bar{a}_r}{r^2}$$

①

* Gauss Theorem or Divergence Theorem



$$\text{outflow} = \int_V \nabla \cdot \vec{J} \, dV = \oint_S \vec{J} \cdot \vec{ds}$$

* Green's first Identity

Let $\vec{J} = \Phi \nabla \psi$ for any scalar fields Φ and ψ

$$\int_V \nabla \cdot (\Phi \nabla \psi) \, dV = \oint_S \Phi \nabla \psi \cdot \vec{ds}$$

$$\int_V \Phi \nabla^2 \psi \, dV + \int_V \nabla \Phi \cdot \nabla \psi \, dV = \oint_S \Phi \nabla \psi \cdot \vec{ds} \quad \text{--- ①}$$

* Green's Second Identity (12)

We exchange Φ and ψ .

$$\int_V \psi \nabla^2 \Phi \, dV + \int_V \nabla \psi \cdot \nabla \Phi \, dV = \oint_S \psi \nabla \Phi \cdot \bar{d}\mathbf{s}$$

From ① - ②,

— ②

$$\int_V (\Phi \nabla^2 \psi - \psi \nabla^2 \Phi) \, dV = \oint_S (\Phi \nabla \psi - \psi \nabla \Phi) \cdot \bar{d}\mathbf{s}$$

\Rightarrow Green's Theorem

"Source inside" \leftrightarrow "potential on surface"