Plonsey & Barr Bioelectricity 3rd ed.

1. Vector Analysis

$\begin{cases} \text{scalar} & : \text{ only magnitude} \\ \text{vector} & : \text{ magnitude \& direction} \end{cases}$

$\begin{cases} \text{scalar field} & : \text{ scalar function of position} \\ \text{vector field} & : \text{ vector} \quad '' \quad '' \end{cases}$

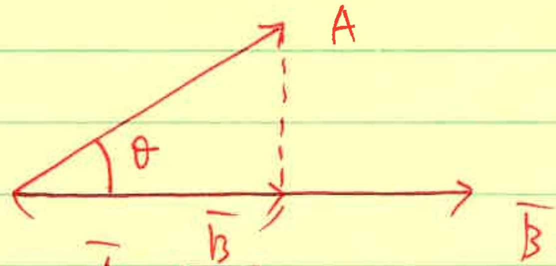
Sum : $\bar{C} = \bar{A} + \bar{B}$

~~Scalar product~~ [scaling] : $\bar{B} = m\bar{A}$

Unit vector : $\hat{a} = \dfrac{\bar{A}}{|A|}$

Dot product (or Scalar product)

$$\bar{A} \cdot \bar{B} = |\bar{A}|\,|\bar{B}| \cos\theta = AB\cos\theta$$
$$= \bar{B} \cdot \bar{A} = A_x B_x + A_y B_y + A_z B_z$$



$\underbrace{\bar{A} \cdot \dfrac{\bar{B}}{|B|}}$ component of $\bar{A}$ along $\bar{B}$

or

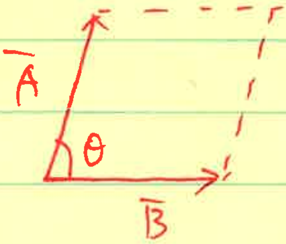
projection of $\bar{A}$ along $\bar{B}$

(Note : $\theta = 90°$ and $\theta = 0°$)

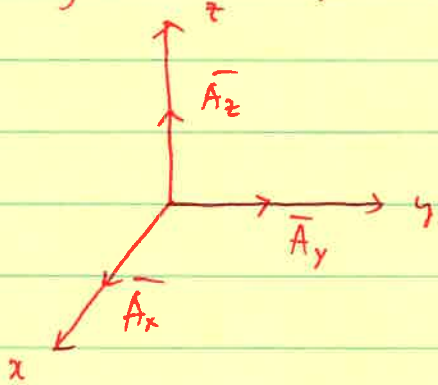
Cross product (or Vector product)

$$\bar{C} = \bar{A} \times \bar{B}$$

$$|\bar{C}| = |\bar{A}|\,|\bar{B}| \sin\theta \quad : \text{ area of parallelogram}$$



Direction of $\bar{C}$ follows RH Rule ($\bar{C}$ $\otimes$ "into the page")

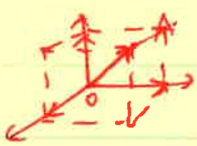


$$\bar{A}_x = \bar{A}_y \times \bar{A}_z$$

$$\bar{A} \times \bar{B} = (A_y B_z - A_z B_y)\,\bar{a}_x + (A_z B_x - A_x B_z)\,\bar{a}_y + (A_x B_y - A_y B_x)\,\bar{a}_z$$

$$= \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Components of Vector : $\bar{A} = A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z$



* Gradient

$\Phi(x, y, z)$: scalar field

- single-valued
- continuous
- differentiable

$\Phi(x, y, z) = C$ defines a <u>surface</u> (equipotential surface)

cf. contour map

$$P_1(x, y, z) \xrightarrow{\ \bar{dl}\ } P_2(x+dx,\ y+dy,\ z+dz)$$

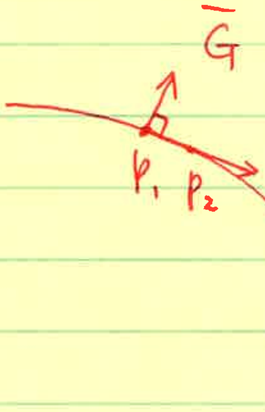
$$\bar{dl} = dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z$$

$$\begin{aligned} d\Phi &= \Phi(x+dx,\ y+dy,\ z+dz) - \Phi(x, y, z) \\ &= \frac{\partial \Phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy + \frac{\partial \Phi}{\partial z} dz \\ &= \Phi(P_2) - \Phi(P_1) = \text{total derivative at } P_1 \end{aligned}$$

$$\bar{G} \triangleq \frac{\partial \Phi}{\partial x}\bar{a}_x + \frac{\partial \Phi}{\partial y}\bar{a}_y + \frac{\partial \Phi}{\partial z}\bar{a}_z \quad : \text{Gradient of } \Phi$$

$$d\Phi = \bar{G} \cdot \bar{dl}$$

If



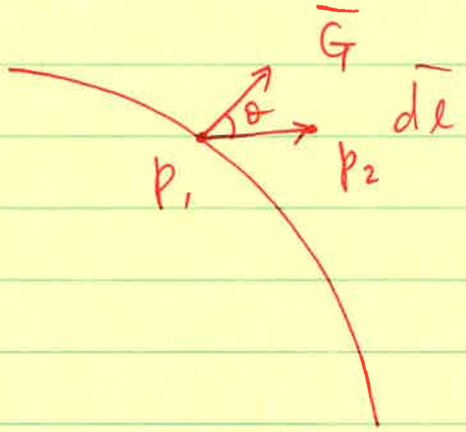
$\bar{dl}$: tangential $(\because P_1 \simeq P_2)$

$\Phi(P_1) = \Phi(P_2) = C_1$,

$d\Phi = 0 = \bar{G} \cdot \bar{dl} \Rightarrow \bar{G} \perp \bar{dl}$

$\Rightarrow \bar{G}$ is normal to surface C_1 at P_1

In general,



$\Phi(x, y, z) = C_1$

$$d\Phi = \bar{G} \cdot \bar{dl}$$

$$= |\bar{G}||\bar{dl}| \cos\theta$$

$$\Rightarrow \boxed{\frac{d\Phi}{dl} = G \cos\theta}$$

Derivative of G in the direction l

(Directional derivative)

When $\theta = 0$, $\bar{dl} = \bar{n}$: outward unit normal vector

$$G = \frac{d\Phi}{dn}$$

$\bar{G} = G \cdot \bar{n}$: Gradient

Direction $(\bar{n})$ is normal to the iso-surface
G (magnitude) is the maximum rate of increase in Φ

cf, Steepest descent method

* Del operator

$$\nabla \triangleq \frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z$$

$$\bar{G} = \nabla \Phi = \frac{\partial \Phi}{\partial x}\bar{a}_x + \frac{\partial \Phi}{\partial y}\bar{a}_y + \frac{\partial \Phi}{\partial z}\bar{a}_z$$

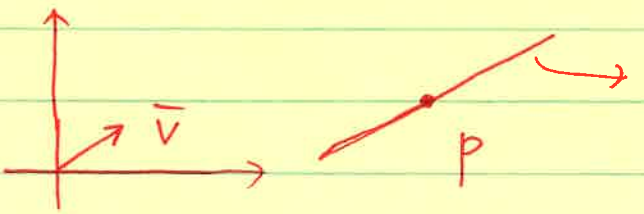
$$|\bar{G}| = G = |\nabla \Phi| = \sqrt{\left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2}$$

Φ : scalar, potential, effort variable

$\nabla \Phi$: vector, flow, flow variable

* Directional Derivative and Gradient

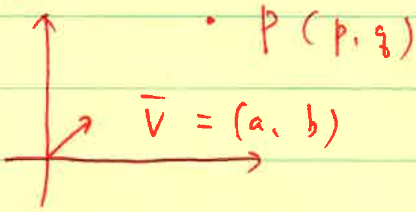
f : differentiable , p : point



$p + t\bar{v}$, $\|\bar{v}\| = 1$
(line)

$$\frac{d(p + t\bar{v})}{dt} = \bar{v}$$

Ex)



$$p + t\bar{v} = (p + ta,\ q + tb)$$

$$\frac{d(p + t\bar{v})}{dt} = (a, b) = \bar{v}$$

Let $g(t) = f(p + t\bar{v})$: change of f along the line $p + t\bar{v}$

$$\frac{df(p+t\bar{v})}{dt} = \nabla f(p + t\bar{v}) \cdot \frac{d(p+t\bar{v})}{dt} = \nabla f(p+t\bar{v}) \cdot \bar{v}$$

(chain rule)

If $t = 0$, $\nabla f(p) \cdot \bar{v} = D_{\bar{v}} f(p)$

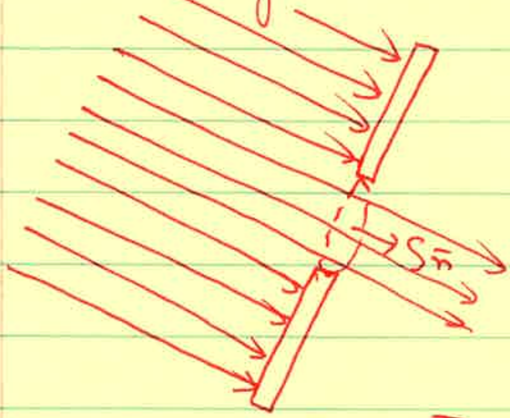
$\Rightarrow$ Directional derivative of f in the direction of $\bar{v}$ at p

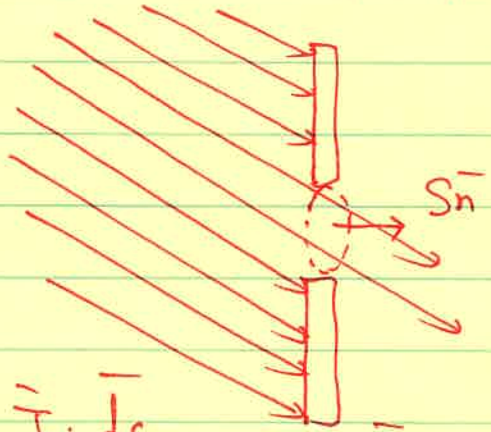
If $\nabla f(p) \,/\!/\, \bar{v}$, $|D_{\bar{v}} f(p)|$ is maximal.

$\Rightarrow$ Direction of $\nabla f(p)$ is the steepest direction of f at p.

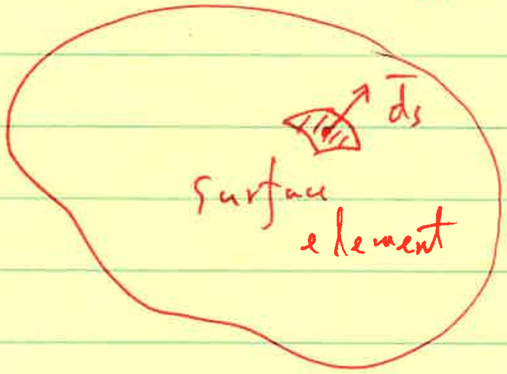
vector field

* Divergence : Assume $\bar{J}(x, y, z)$

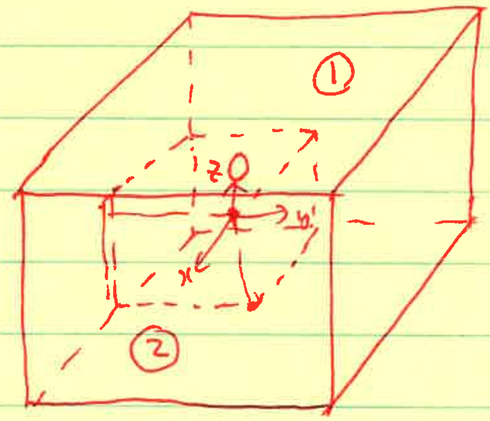




$$I = \oint_S \bar{J} \cdot \bar{ds} = \bar{J} \cdot S\bar{n} = JS\cos\theta$$



$$\left\{ \begin{array}{l} \bar{ds} = ds \cdot \bar{n} \\ \bar{n} : \text{outward unit normal vector} \end{array} \right.$$



$o(x, y, z)$

$$\text{Outflow}\Big|_{2} = \left[J_x + \frac{1}{2} \frac{\partial J_x}{\partial x} dx \right] dy\, dz$$

$$\text{Outflow}\Big|_{①} = \left[J_x - \frac{1}{2}\frac{\partial J_x}{\partial x}dx \right](-dy\,dz)$$

$$\oint_S \bar{J}\cdot\bar{ds} = \left[\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}\right] dx\,dy\,dz$$

$$\operatorname{div}\bar{J} \triangleq \lim_{V\to 0}\frac{\oint \bar{J}\cdot\bar{ds}}{V}$$

$$= \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}$$

$$= \nabla\cdot\bar{J}$$

$$\begin{cases} \operatorname{div}\bar{J} = 0 & : \text{source (or sink) free} \\ & \quad - \text{continuity of current} \\ & \quad - \text{conservation of charge} \\ \operatorname{div}\bar{J} > 0 & : \text{source} \\ \operatorname{div}\bar{J} < 0 & : \text{sink} \end{cases}$$

* Laplacian

Φ : potential

$\nabla\cdot\Phi$: flow

$$\nabla\cdot\nabla\Phi = \nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}$$

In a source-free region,

$$\nabla^2 \Phi = 0 \quad : \text{Laplace equation}$$

cf. Measure Φ and compute $\nabla^2 \Phi$.
If $\nabla^2 \Phi > 0$ at P, source is there.
If $\nabla^2 \Phi < 0$ at Q, sink is there.

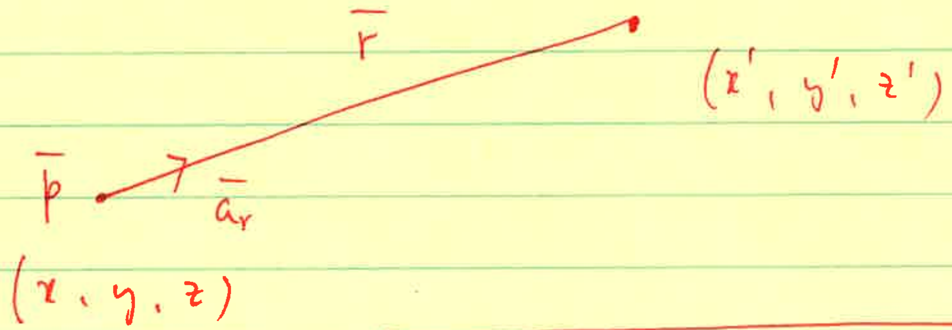
* Note

$$\nabla \cdot (\Phi \bar{A}) = \bar{A} \cdot \nabla \Phi + \Phi \, \nabla \cdot \bar{A}$$

$$\nabla (\Phi \Psi) = \Phi \nabla \Psi + \Psi \nabla \Phi$$

$$\nabla^2 r = 0 \quad \text{where } r = \sqrt{x^2 + y^2 + z^2}$$

* Source and field points



$$r = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$= r(x, y, z, x', y', z')$$

$$\bar{a}_r = \frac{(x'-x)\bar{a}_x + (y'-y)\bar{a}_y + (z'-z)\bar{a}_z}{r}$$

$$\nabla\left(\frac{1}{r}\right) = \frac{\partial}{\partial x}\left(\frac{1}{r}\right)\bar{a}_x + \frac{\partial}{\partial y}\left(\frac{1}{r}\right)\bar{a}_y + \frac{\partial}{\partial z}\left(\frac{1}{r}\right)\bar{a}_z$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{\partial}{\partial x}\left\{(x-x')^2+(y-y')^2+(z-z')^2\right\}^{-\frac{1}{2}}$$

$$= -\frac{1}{2}\left\{(x-x')^2+(y-y')^2+(z-z')^2\right\}^{-\frac{3}{2}} \times 2(x-x')$$

$$= -\frac{x-x'}{r^3}$$

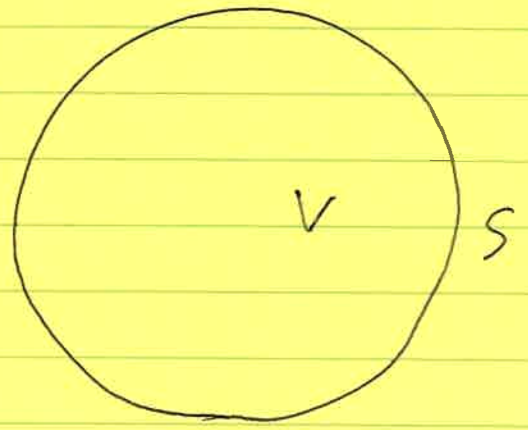
$$\nabla\left(\frac{1}{r}\right) = -\frac{(x-x')\bar{a}_x + (y-y')\bar{a}_y + (z-z')\bar{a}_z}{r^3}$$

$$= \left(-\frac{1}{r^2}\right) \times \left(-\bar{a}_r\right)$$

$$= \frac{\bar{a}_r}{r^2}$$

$$\nabla'\left(\frac{1}{r}\right) = -\nabla\left(\frac{1}{r}\right) = -\frac{\bar{a}_r}{r^2}$$

* Gauss Theorem or Divergence Theorem



$$\text{outflow} = \int_V \nabla \cdot \bar{J} \, dv = \oint_S \bar{J} \cdot \bar{ds}$$

* Green's First Identity

Let $\bar{J} = \Phi \nabla \Psi$ for any scalar fields Φ and Ψ

$$\int_V \nabla \cdot (\Phi \nabla \Psi) \, dv = \oint_S \Phi \nabla \Psi \cdot \bar{ds}$$

$$\int_V \Phi \nabla^2 \Psi \, dv + \int_V \nabla \Phi \cdot \nabla \Psi \, dv = \oint_S \Phi \nabla \Psi \cdot \bar{ds} \quad \text{— (1)}$$

* Green's Second Identity

We exchange Φ and ψ.

$$\int_V \psi \nabla^2 \Phi \, dv + \int_v \nabla\psi \cdot \nabla\Phi \, dv = \oint_S \psi \nabla \Phi \cdot \overline{ds} \quad \text{— ②}$$

From ① − ②,

$$\int_V \left(\Phi \nabla^2 \psi - \psi \nabla^2 \Phi \right) dv = \oint_S \left(\Phi \nabla \psi - \psi \nabla \Phi \right) \cdot \overline{ds}$$

$\Rightarrow$ Green's Theorem

"Source inside" $\longleftrightarrow$ "potential on surface"