Poisson Equation and Laplace Equation

- Elliptic Partial Differential Equation
- Boundary Value Problem

$$\bar{J} = \bar{J}^i + \sigma \bar{E} = \bar{J}^i - \sigma \nabla \phi$$

$$\nabla \cdot \bar{J} = \nabla \cdot \bar{J}^i - \nabla \cdot (\sigma \nabla \phi) = 0$$

$$\nabla \cdot (\sigma \nabla \phi) = \nabla \cdot \bar{J}^i$$

for a homogeneous domain,

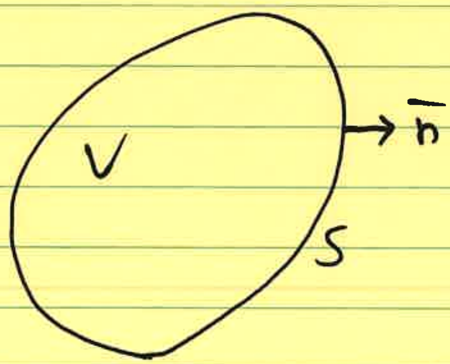
$$\sigma \nabla^2 \phi = \nabla \cdot \bar{J}^i$$

From the divergence theorem,

$$\int_v \nabla \cdot (\psi \nabla \phi)\, dv = \int_s (\psi \nabla \phi) \cdot \bar{ds}$$

$$\nabla \cdot (\psi \nabla \phi) = \nabla \psi \cdot \nabla \phi + \psi \nabla^2 \phi$$

$$\nabla \phi \cdot \bar{ds} = \nabla \phi \cdot \bar{n}\, ds = \frac{\partial \phi}{\partial n} ds$$



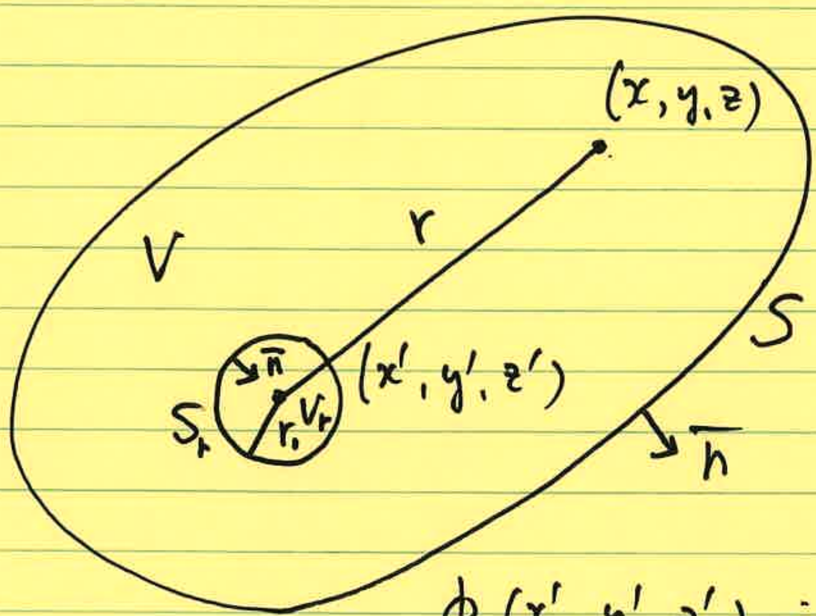
$$\int_v \nabla\psi\cdot\nabla\phi\, dv + \int_v \psi\nabla^2\phi\, dv = \int_s \psi\frac{\partial\phi}{\partial n}\, ds$$

Exchanging ψ and ϕ,

$$\int_v \nabla\phi\cdot\nabla\psi\, dv + \int_v \phi\nabla^2\psi\, dv = \int_s \phi\frac{\partial\psi}{\partial n}\, ds$$

$$\int_v \left(\psi\nabla^2\phi - \phi\nabla^2\psi\right) dv = \int_s \left(\psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n}\right) ds$$

$\Rightarrow$ Green's Theorem



$\bar{J}^i(x, y, z)$: impressed current at source point

$\phi(x', y', z')$: potential at field point

$$r = \sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2}$$

$\psi = \dfrac{1}{r}$: singular at $r = 0$

V_r is a sphere defined to avoid this singularity

Inside $V \setminus V_r$, $\nabla^2 \phi = \nabla^2 \frac{1}{r} = 0$.

(See note on monopole field)

$$\int_v \frac{\nabla^2 \phi}{r} dv = \int_{S+S_r} \left[\frac{1}{r} \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n} \frac{1}{r} \right] ds$$

On S_r, $\frac{\partial \phi}{\partial n} = - \frac{\partial \phi}{\partial r}$

$$\left. \frac{\partial}{\partial n} \frac{1}{r} \right|_{r=r_1} = - \left. \frac{\partial}{\partial r} \frac{1}{r} \right|_{r=r_1} = \frac{1}{r_1^2}$$

$$\int_{S_r (r=r_1)} \left[\frac{1}{r} \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n} \frac{1}{r} \right] ds$$

$$= - \frac{1}{r_1} \int_{S_r} \frac{\partial \phi}{\partial r} ds - \frac{1}{r_1^2} \int_{S_r} \phi \, ds$$

$$= - \frac{1}{r_1} \left\langle \frac{\partial \phi}{\partial r} \right\rangle 4\pi r_1^2 - \frac{1}{r_1^2} \langle \phi \rangle 4\pi r_1^2$$

$$= - 4\pi r_1 \left\langle \frac{\partial \phi}{\partial r} \right\rangle - 4\pi \langle \phi \rangle$$

$$\lim_{r_1 \to 0} \int_{S_r} \left[\frac{1}{r} \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n} \frac{1}{r} \right] ds = - 4\pi \phi(x', y', z')$$

$$\int_v \frac{\nabla^2 \phi}{r} dv = - 4\pi \phi + \int_S \left[\frac{1}{r} \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n} \frac{1}{r} \right] ds$$

$$\phi(x', y', z') = -\frac{1}{4\pi}\int_v \frac{\nabla^2\phi}{r}\,dv + \frac{1}{4\pi}\int_s \left[\frac{1}{r}\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}\frac{1}{r}\right]ds$$

Since $\sigma\nabla^2\phi = \nabla\cdot\bar{J}^i$,

$$\phi = -\frac{1}{4\pi\sigma}\int_v \frac{\nabla\cdot\bar{J}^i}{r}\,dv + \frac{1}{4\pi}\int_s \left[\frac{1}{r}\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}\frac{1}{r}\right]ds$$

Consider $\nabla\cdot\left(\frac{\bar{J}^i}{r}\right) = \nabla\left(\frac{1}{r}\right)\cdot\bar{J}^i + \frac{1}{r}\nabla\cdot\bar{J}^i$

From the divergence theorem,

$$\int_v \nabla\cdot\left(\frac{\bar{J}^i}{r}\right)dv = \int_s \frac{\bar{J}^i}{r}\cdot\bar{ds} = 0$$

($\because$ There is no source on the boundary)

$$\int_v \frac{\nabla\cdot\bar{J}^i}{r}\,dv = -\int_v \bar{J}^i\cdot\nabla\left(\frac{1}{r}\right)dv$$

$$\phi = \frac{1}{4\pi\sigma}\int_v \bar{J}^i\cdot\nabla\left(\frac{1}{r}\right)dv + \frac{1}{4\pi}\int_s \left[\frac{1}{r}\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}\frac{1}{r}\right]ds$$

$$= -\frac{1}{4\pi\sigma}\int_v \frac{\nabla\cdot\bar{J}^i}{r}\,dv + \frac{1}{4\pi}\int_s \left(\frac{1}{r}\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}\frac{1}{r}\right)ds$$

* Laplace Equation

If $\nabla \cdot \bar{J}^i = 0$ in V, $\nabla^2 \phi = 0$.

$$\phi = \frac{1}{4\pi} \int_S \left[\frac{1}{r} \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n} \frac{1}{r} \right] ds$$ is a

homogeneous solution of $\nabla^2 \phi = 0$ in V.

ϕ is determined by ϕ and $\frac{\partial \phi}{\partial n}$ on the boundary S.

* Poisson Equation

If $\nabla \cdot \bar{J}^i \neq 0$ in V, $\nabla^2 \phi = \frac{\nabla \cdot \bar{J}^i}{\sigma}$.

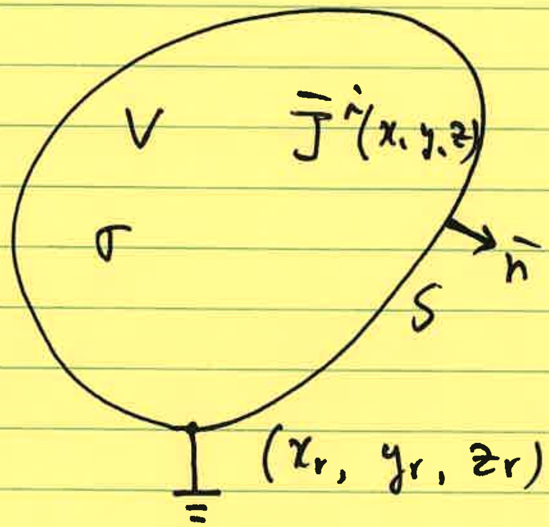
$$\phi = -\frac{1}{4\pi\sigma} \int_V \frac{\nabla \cdot \bar{J}^i}{r} dv$$ is a

particular solution of $\nabla^2 \phi = \frac{\nabla \cdot \bar{J}^i}{\sigma}$.

The general solution of $\nabla^2 \phi = \frac{\nabla \cdot \bar{J}^i}{\sigma}$ is

$$\phi = -\frac{1}{4\pi\sigma} \int_V \frac{\nabla \cdot \bar{J}^i}{r} dv + \frac{1}{4\pi} \int_S \left[\frac{1}{r} \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n} \frac{1}{r} \right] ds$$

* Boundary Value Problem I



$\sigma = 0$

$\Downarrow$

$\frac{\partial \phi}{\partial n} = 0$

$\phi(x_r, y_r, z_r) = 0$

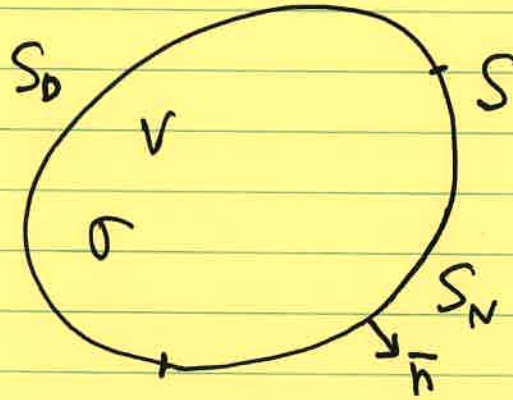
$$\begin{cases} \nabla^2 \phi = \dfrac{\nabla \cdot \bar{J}^i}{\sigma} & \text{in } V \\ \phi = 0 & \text{at } (x_r, y_r, z_r) \\ \dfrac{\partial \phi}{\partial n} = 0 & \text{on } S \setminus (x_r, y_r, z_r) \end{cases}$$

- Dirichlet Boundary Condition
- Neumann Boundary Condition

$$\phi = -\frac{1}{4\pi\sigma} \int_v \frac{\nabla \cdot \bar{J}^i}{r} \, dv = \frac{1}{4\pi\sigma} \int_v \bar{J}^i \cdot \nabla\left(\frac{1}{r}\right) dv$$

If $\nabla \cdot \bar{J}^i = 0$ in V, $\phi = 0$.

* Boundary Value Problem II



$\sigma = 0$

$S_D \cup S_N = S$

$$
\begin{cases}
\nabla^2 \phi = \dfrac{\nabla \cdot \bar{J}^i}{\sigma} & \text{in } V \\
\phi = [\phi] & \text{on } S_D \\
\dfrac{\partial \phi}{\partial n} = \left[\dfrac{\partial \phi}{\partial n}\right] & \text{on } S_N
\end{cases}
$$

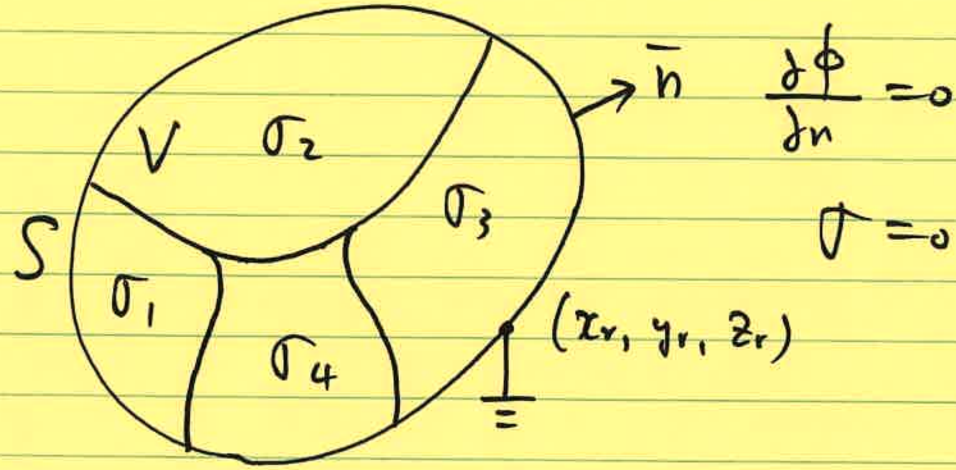
$$
\phi = \frac{1}{4\pi\sigma}\int_V \bar{J}^i \cdot \nabla\left(\frac{1}{r}\right) dv + \frac{1}{4\pi}\int_S \left[\frac{1}{r}\frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n}\frac{1}{r}\right] ds
$$

* Boundary Value Problem III

$$
\begin{cases}
\nabla^2 \phi = 0 & \text{in } V \\
\phi = [\phi] & \text{on } S_D \\
\dfrac{\partial \phi}{\partial n} = \left[\dfrac{\partial \phi}{\partial n}\right] & \text{on } S_N
\end{cases}
$$

$$
\phi = \frac{1}{4\pi}\int_S \left[\frac{1}{r}\frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n}\frac{1}{r}\right] ds
$$

* Effects of inhomogeneous conductivity



$$\begin{cases} \nabla \cdot (\sigma \nabla \phi) = \nabla \cdot \bar{J}^i \\ \phi = 0 \quad \text{at } (x_r, y_r, z_r) \\ \frac{\partial \phi}{\partial n} = 0 \quad \text{on } S \setminus (x_r, y_r, z_r) \end{cases}$$

$$\phi = \frac{1}{4\pi\sigma} \int_v \bar{J}^i \cdot \nabla\left(\frac{1}{r}\right) dv$$

$$+ \sum_j \int_{S_j} (\sigma_j'' - \sigma_j') \phi \, \bar{n}_j \cdot \nabla\left(\frac{1}{r}\right) ds_j$$