

# Poisson Equation and Laplace Equation ①

- Elliptic Partial Differential Equation
- Boundary Value Problem

$$\bar{J} = \bar{J}^i + \sigma \bar{E} = \bar{J}^i - \sigma \nabla \phi$$

$$\nabla \cdot \bar{J} = \nabla \cdot \bar{J}^i - \nabla \cdot (\sigma \nabla \phi) = 0$$

$$\nabla \cdot (\sigma \nabla \phi) = \nabla \cdot \bar{J}^i$$

for a homogeneous domain,

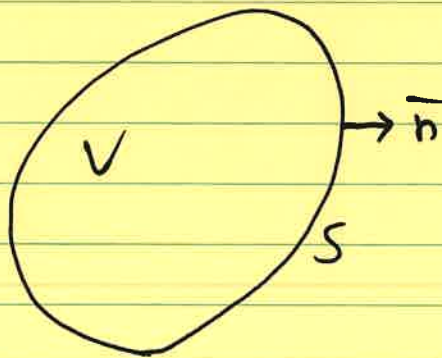
$$\sigma \nabla^2 \phi = \nabla \cdot \bar{J}^i$$

From the divergence theorem,

$$\int_V \nabla \cdot (\psi \nabla \phi) dV = \int_S (\psi \nabla \phi) \cdot \bar{dS}$$

$$\nabla \cdot (\psi \nabla \phi) = \nabla \psi \cdot \nabla \phi + \psi \nabla^2 \phi$$

$$\nabla \phi \cdot \bar{dS} = \nabla \phi \cdot \bar{n} dS = \frac{\partial \phi}{\partial n} dS$$



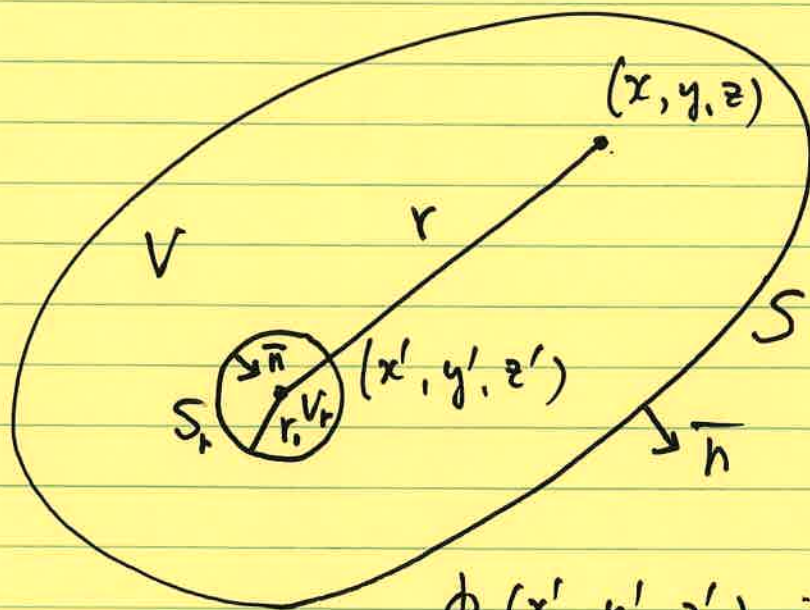
$$\int_V \nabla\psi \cdot \nabla\phi \, dV + \int_V \psi \nabla^2\phi \, dV = \int_S \psi \frac{\partial\phi}{\partial n} \, dS \quad (2)$$

Exchanging  $\psi$  and  $\phi$ ,

$$\int_V \nabla\phi \cdot \nabla\psi \, dV + \int_V \phi \nabla^2\psi \, dV = \int_S \phi \frac{\partial\psi}{\partial n} \, dS$$

$$\int_V (\psi \nabla^2\phi - \phi \nabla^2\psi) \, dV = \int_S (\psi \frac{\partial\phi}{\partial n} - \phi \frac{\partial\psi}{\partial n}) \, dS$$

$\Rightarrow$  Green's Theorem



$\vec{J}^i(x, y, z)$ :  
impressed current  
at source point

$\phi(x', y', z')$ : potential at  
field point

$$r = \sqrt{(x' - x)^2 + (y' - y)^2 + (z' - z)^2}$$

$$\psi = \frac{1}{r} \quad : \quad \text{Singular at } r = 0$$

$V_r$  is a sphere defined to avoid  
this singularity

$$\text{Inside } V \setminus V_r, \quad \nabla^2 \psi = \nabla^2 \frac{1}{r} = 0. \quad (3)$$

(See note on monopole field)

$$\int_V \frac{\nabla^2 \phi}{r} dV = \int_{S+S_r} \left[ \frac{1}{r} \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n} \frac{1}{r} \right] dS$$

$$\text{On } S_r, \quad \frac{\partial \phi}{\partial n} = - \frac{\partial \phi}{\partial r}$$

$$\frac{\partial}{\partial n} \frac{1}{r} \Big|_{r=r_1} = - \frac{\partial}{\partial r} \frac{1}{r} \Big|_{r=r_1} = \frac{1}{r_1^2}$$

$$\begin{aligned} & \int_{S_r(r=r_1)} \left[ \frac{1}{r} \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n} \frac{1}{r} \right] dS \\ &= - \frac{1}{r_1} \int_{S_r} \frac{\partial \phi}{\partial r} dS - \frac{1}{r_1^2} \int_{S_r} \phi dS \\ &= - \frac{1}{r_1} \left\langle \frac{\partial \phi}{\partial r} \right\rangle 4\pi r_1^2 - \frac{1}{r_1^2} \langle \phi \rangle 4\pi r_1^2 \\ &= - 4\pi r_1 \left\langle \frac{\partial \phi}{\partial r} \right\rangle - 4\pi \langle \phi \rangle \end{aligned}$$

$$\lim_{r_1 \rightarrow 0} \int_{S_r} \left[ \frac{1}{r} \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n} \frac{1}{r} \right] dS = - 4\pi \phi(x', y', z')$$

$$\int_V \frac{\nabla^2 \phi}{r} dV = - 4\pi \phi + \int_S \left[ \frac{1}{r} \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n} \frac{1}{r} \right] dS$$

$$\phi(x', y', z') = -\frac{1}{4\pi} \int_V \frac{\nabla^2 \phi}{r} d\tau + \frac{1}{4\pi} \int_S \left[ \frac{1}{r} \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n} \left( \frac{1}{r} \right) \right] ds \quad (4)$$

Since  $\sigma \nabla^2 \phi = \nabla \cdot \bar{J}^i$ ,

$$\star \phi = -\frac{1}{4\pi\sigma} \int_V \frac{\nabla \cdot \bar{J}^i}{r} d\tau + \frac{1}{4\pi} \int_S \left[ \frac{1}{r} \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n} \left( \frac{1}{r} \right) \right] ds$$

Consider  $\nabla \cdot \left( \frac{\bar{J}^i}{r} \right) = \nabla \left( \frac{1}{r} \right) \cdot \bar{J}^i + \frac{1}{r} \nabla \cdot \bar{J}^i$

From the divergence theorem,

$$\int_V \nabla \cdot \left( \frac{\bar{J}^i}{r} \right) d\tau = \int_S \frac{\bar{J}^i}{r} \cdot \bar{d}s = 0$$

( $\because$  There is no source on the boundary)

$$\int_V \frac{\nabla \cdot \bar{J}^i}{r} d\tau = - \int_V \bar{J}^i \cdot \nabla \left( \frac{1}{r} \right) d\tau$$

$$\begin{aligned} \star \phi &= \frac{1}{4\pi\sigma} \int_V \bar{J}^i \cdot \nabla \left( \frac{1}{r} \right) d\tau + \frac{1}{4\pi} \int_S \left[ \frac{1}{r} \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n} \left( \frac{1}{r} \right) \right] ds \\ &= -\frac{1}{4\pi\sigma} \int_V \frac{\nabla \cdot \bar{J}^i}{r} d\tau + \frac{1}{4\pi} \int_S \left[ \frac{1}{r} \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n} \left( \frac{1}{r} \right) \right] ds \end{aligned}$$

### \* Laplace Equation

If  $\nabla \cdot \vec{J}^i = 0$  in  $V$ ,  $\nabla^2 \phi = 0$ .

$$\phi = \frac{1}{4\pi} \int_S \left[ \frac{1}{r} \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n} \frac{1}{r} \right] dS \quad \text{is a}$$

homogeneous solution of  $\nabla^2 \phi = 0$  in  $V$ .

$\phi$  is determined by  $\phi$  and  $\frac{\partial \phi}{\partial n}$  on the boundary  $S$ .

### \* Poisson Equation

If  $\nabla \cdot \vec{J}^i \neq 0$  in  $V$ ,  $\nabla^2 \phi = \frac{\nabla \cdot \vec{J}^i}{\sigma}$ .

$$\phi = -\frac{1}{4\pi\sigma} \int_V \frac{\nabla \cdot \vec{J}^i}{r} dv \quad \text{is a}$$

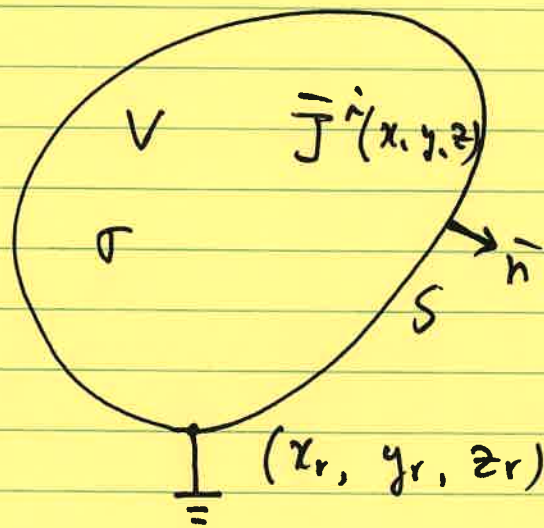
particular solution of  $\nabla^2 \phi = \frac{\nabla \cdot \vec{J}^i}{\sigma}$ .

The general solution of  $\nabla^2 \phi = \frac{\nabla \cdot \vec{J}^i}{\sigma}$  is

$$\phi = -\frac{1}{4\pi\sigma} \int_V \frac{\nabla \cdot \vec{J}^i}{r} dv + \frac{1}{4\pi} \int_S \left[ \frac{1}{r} \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n} \frac{1}{r} \right] dS$$

# \* Boundary Value Problem I

(6)



$$\sigma = 0$$

$$\Downarrow$$

$$\frac{\partial \phi}{\partial n} = 0$$

$$\phi(x_r, y_r, z_r) = 0$$

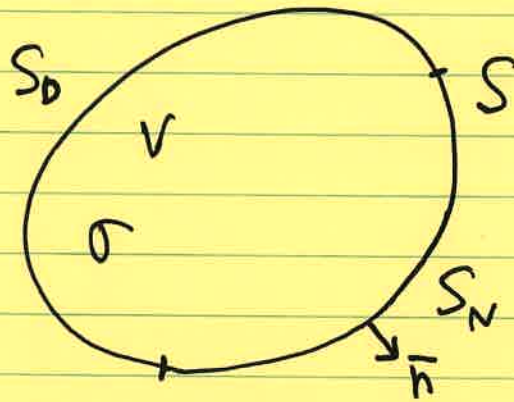
$$\left\{ \begin{array}{l} \nabla^2 \phi = \frac{\sigma \cdot \vec{J}^i}{\sigma} \quad \text{in } V \\ \phi = 0 \quad \text{at } (x_r, y_r, z_r) \\ \frac{\partial \phi}{\partial n} = 0 \quad \text{on } S \setminus (x_r, y_r, z_r) \end{array} \right.$$

- Dirichlet Boundary Condition
- Neumann Boundary Condition

$$\phi = -\frac{1}{4\pi\sigma} \int_V \frac{\nabla \cdot \vec{J}^i}{r} d\tau = \frac{1}{4\pi\sigma} \int_V \vec{J}^i \cdot \nabla\left(\frac{1}{r}\right) d\tau$$

If  $\nabla \cdot \vec{J}^i = 0$  in  $V$ ,  $\phi = 0$ .

## \* Boundary Value Problem II



$$\nabla \cdot \vec{j} = 0$$

$$S_D \cup S_N = S$$

$$\begin{cases} \nabla^2 \phi = \frac{\nabla \cdot \vec{j}}{\sigma} & \text{in } V \\ \phi = [\phi] & \text{on } S_D \\ \frac{\partial \phi}{\partial n} = \left[ \frac{\partial \phi}{\partial n} \right] & \text{on } S_N \end{cases}$$

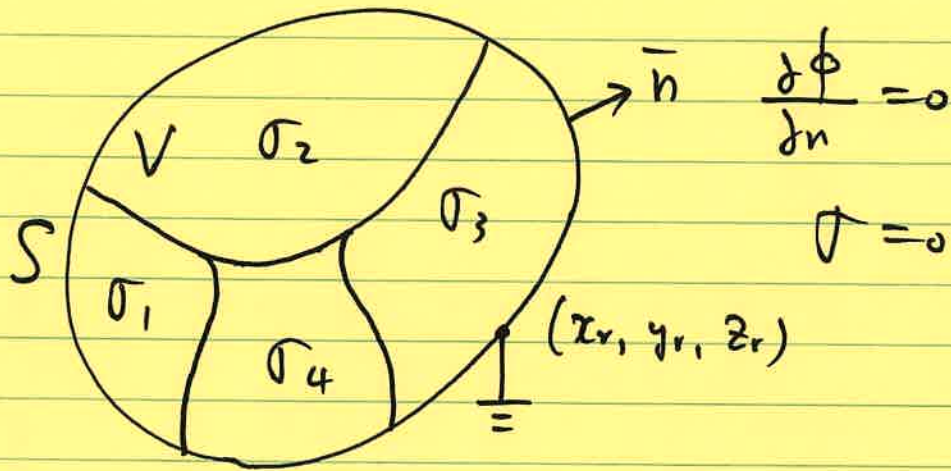
$$\phi = \frac{1}{4\pi\sigma} \int_V \vec{j} \cdot \nabla \left( \frac{1}{r} \right) dV + \frac{1}{4\pi} \int_S \left[ \frac{1}{r} \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n} \frac{1}{r} \right] dS$$

## \* Boundary Value Problem III

$$\begin{cases} \nabla^2 \phi = 0 & \text{in } V \\ \phi = [\phi] & \text{on } S_D \\ \frac{\partial \phi}{\partial n} = \left[ \frac{\partial \phi}{\partial n} \right] & \text{on } S_N \end{cases}$$

$$\phi = \frac{1}{4\pi} \int_S \left[ \frac{1}{r} \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n} \frac{1}{r} \right] dS$$

# \* Effects of inhomogeneous conductivity (8)



$$\left\{ \begin{array}{l} \nabla \cdot (\sigma \nabla \phi) = \nabla \cdot \bar{J}^i \\ \phi = 0 \quad \text{at } (x_r, y_r, z_r) \\ \frac{\partial \phi}{\partial n} = 0 \quad \text{on } S \setminus (x_r, y_r, z_r) \end{array} \right.$$

$$\phi = \frac{1}{4\pi\sigma} \int_V \bar{J}^i \cdot \nabla \left( \frac{1}{r} \right) dV + \sum_j \int_{S_j} (\sigma_j'' - \sigma_j') \phi \bar{n}_j \cdot \nabla \left( \frac{1}{r} \right) dS_j$$