1. Signals and Systems

Signal

(1) Function of time, space, etc

(2) Contains information

(3) *f* is a whole signal and *f*(*t*) is the value of signal at time *t*.

(4) Domain of signal: *t*'s for which it is defined

Continuous-time signal

Domain of signal is **R**.

Discrete-time signal

Domain of signal is **I**.

Unit and dimension of signal

Measure of signal

Properties of signal

Power and energy

	Continuous-time	*Discrete-time*
Instantaneous power	$p(t) = x^2(t)$	$p[n] = x^2[n]$
Average power	$P = \frac{1}{t_2 - t_1}\int_{t_2}^{t_1} x^2(t)dt$ $P_\infty \overset{\Delta}{=} \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x^2(t)dt$	$P = \frac{1}{n_2 - n_1 + 1}\sum_{n=n_1}^{n_2} x^2[n]$ $P_\infty \overset{\Delta}{=} \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} x^2[n]$
Energy	$E = \int_{t_2}^{t_1} x^2(t)dt$ $E_\infty \overset{\Delta}{=} \lim_{T\to\infty} \int_{-T}^{T} x^2(t)dt$	$E = \sum_{n=n_1}^{n_2} x^2[n]$ $E_\infty \overset{\Delta}{=} \lim_{N\to\infty} \sum_{n=-N}^{N} x^2[n]$

(1) Signal with finite energy signal: $E_\infty < \infty$ and $P_\infty = 0$, find an example.

(2) Signal with finite average power: $E_\infty = \infty$ and $P_\infty < \infty$, find an example.

Transformations of the independent variable

	Continuous-time	*Discrete-time*
Time shift	$x(t-t_0)$	$x[n-n_0]$
Time reversal	$x(-t)$	$x[-n]$
Time scaling	$x(\alpha t)$	$x[\alpha n]$
Combination	$x(\alpha t+\beta)$	$x[\alpha n+\beta]$

Characteristics of signal

	Continuous-time	*Discrete-time*
Periodic	$x(t)=x(t+T)$ (fundamental) period, T	$x[n]=x[n+N]$ (fundamental) period, N
Aperiodic	Not periodic	Not periodic
Even	$x(t)=x(-t)$	$x[n]=x[-n]$
Odd	$x(t)=-x(-t)$	$x[n]=-x[-n]$
Even part	$\mathcal{E}v\{x(t)\}=\frac{1}{2}\{x(t)+x(-t)\}$	$\mathcal{E}v\{x[n]\}=\frac{1}{2}\{x[n]+x[-n]\}$
Odd part	$\mathcal{O}d\{x(t)\}=\frac{1}{2}\{x(t)-x(-t)\}$	$\mathcal{O}d\{x[n]\}=\frac{1}{2}\{x[n]-x[-n]\}$

Continuous-time exponential and sinusoidal signal

(1) Real exponential signal: $x(t)=Ce^{at}$, $C\in\mathbf{R}$ and $a\in\mathbf{R}$

(2) Complex exponential signal: $x(t)=Ce^{at}$, $C=|C|e^{j\theta}\in\mathbf{C}$ and $a=(r+j\omega_0)\in\mathbf{C}$

In this general case, $x(t)=|C|e^{rt}e^{j(\omega_0 t+\theta)}=|C|e^{rt}\{\cos(\omega_0 t+\theta)+j\sin(\omega_0 t+\theta)\}$.

(3) Periodic complex exponential: $x(t)=e^{j\omega_0 t}=x(t+T)=e^{j\omega_0(t+T)}=e^{j\omega_0 T}e^{j\omega_0 t}$

We need $e^{j\omega_0 T}=1$. Therefore,
(a) If $\omega_0=0$, $x(t)=1\ \forall T$.
(b) If $\omega_0\neq 0$, $T_0=T_{\min}=\dfrac{2\pi}{|\omega_0|}$ = fundamental period. Why $T_{\min}$?

(4) Sinusoidal signal: $x(t)=A\cos(\omega_0 t+\phi)$, $T_0=\dfrac{2\pi}{\omega_0}$, $\omega_0=2\pi f_0$

(a) t and T_0: second, sec, or s
(b) ω_0: radians per second or rad/s, fundamental frequency
(c) f_0: cycles per second or Hz (hertz), fundamental frequency

(d) ϕ: radians (what is one radian?)

(e) What are power, energy, average and rms value of $x(t)$?

(f) Plot and describe the voltage signal, $v(t) = 10\cos(2\pi \times 120t + \frac{\pi}{4})$ [V].

(g) Calculate power, energy, average and rms value of $v(t)$ in (f).

(5) Euler's formula: $e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t$

(a) $A\cos(\omega_0 t + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 t} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 t} = A\mathcal{Re}\left\{e^{j(\omega_0 t+\phi)}\right\}$

(b) $A\sin(\omega_0 t + \phi) = \frac{A}{2j}e^{j\phi}e^{j\omega_0 t} - \frac{A}{2j}e^{-j\phi}e^{-j\omega_0 t} = A\mathcal{Im}\left\{e^{j(\omega_0 t+\phi)}\right\}$

(6) Harmonically related complex exponentials: $\phi_k(t) = e^{jk\omega_0 t}, k \in \mathbf{I}$

Note that $k\, f_0 = \frac{k\,\omega_0}{2\pi} = \frac{1}{T_0/k}$. Can you find an inverse relationship?

Discrete-time exponential and sinusoidal signal

(1) Real exponential signal: $x[n] = Ce^{\beta n} = C\alpha^n$, $\alpha = e^{\beta}$, $C \in \mathbf{R}$ and $\alpha, \beta \in \mathbf{R}$

(2) Complex exponential signal: $x[n] = C\alpha^n$, $C = |C|e^{j\theta} \in \mathbf{C}$ and $\alpha = |\alpha|e^{j\omega_0} \in \mathbf{C}$

In this general case, $x[n] = |C||\alpha|^n\{\cos(\omega_0 n + \theta) + j\sin(\omega_0 n + \theta)\}$.

(3) Periodic complex exponential: $x[n] = e^{j\omega_0 n} = x(n+N) = e^{j\omega_0(n+N)} = e^{j\omega_0 N}e^{j\omega_0 n}$

We need $e^{j\omega_0 N} = 1$. Therefore,

(a) When $\omega_0 = 0$, $x[n] = 1\ \forall N$.

(b) When $\omega_0 \neq 0$, $N = m\frac{2\pi}{|\omega_0|}$ = fundamental period if the integer, m has no common factor with the integer, N. Why?

(c) Since $e^{j(\omega_0+2\pi)n} = e^{j\omega_0 n}e^{j2\pi n} = e^{j\omega_0 n}\ \forall n$, $x[n]$ is periodic in ω_0 with period 2π.

(4) Sinusoidal signal: $x[n] = A\cos(\omega_0 n + \phi)$

(a) n: dimensionless

(b) What are power, energy, average and rms value of $x(t)$?

(c) Plot and describe the voltage signal, $v[n]=10\cos(\frac{2\pi}{12}n)$ [V].

(d) Calculate power, energy, average and rms value of $v[n]$ in (c).

(5) Euler's formula: $e^{j\omega_0 n}=\cos\omega_0 n+j\sin\omega_0 n$

(a) $A\cos(\omega_0 n+\phi)=\frac{A}{2}e^{j\phi}e^{j\omega_0 n}+\frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}=A\mathcal{Re}\left\{e^{j(\omega_0 n+\phi)}\right\}$

(b) $A\sin(\omega_0 n+\phi)=\frac{A}{2j}e^{j\phi}e^{j\omega_0 n}-\frac{A}{2j}e^{-j\phi}e^{-j\omega_0 n}=A\mathcal{Im}\left\{e^{j(\omega_0 n+\phi)}\right\}$

(6) Harmonically related complex exponentials: $\phi_k[n]=e^{jk\omega_0 n}, k\in\mathbb{I}$

Note that $\frac{k\,\omega_0}{2\pi}=\frac{1}{N/k}$ and $\phi_k[n]=e^{jk(2\pi/N)n}$. Can you find an inverse relationship?

(7) Revisit the periodicity of $e^{j\omega_0 n}$ w.r.t ω_0.

(a) $e^{j(\omega_0+2\pi)n}=e^{j\omega_0 n}e^{j2\pi n}=e^{j\omega_0 n}\quad\forall n$

(b) $\omega_0=\pi$ is the highest frequency since $e^{j\pi n}=(-1)^n$.

(c) $\omega_0=0,\,2\pi$ are the lowest frequencies since $e^{j2\pi n}=e^{j0n}=1$.

(d) Plot $\cos\omega_0 n$ for $\omega_0=0,\frac{\pi}{8},\frac{\pi}{4},\frac{\pi}{2},\pi,\frac{3\pi}{2},\frac{7\pi}{4},\frac{15\pi}{8},2\pi$. Find a rule.

(8) Revisit the periodicity of $e^{j\omega_0 n}$ w.r.t n.

(a) Since $x[n]=e^{j\omega_0 n}=x(n+N)=e^{j\omega_0(n+N)}=e^{j\omega_0 N}e^{j\omega_0 n}$, we need $e^{j\omega_0 N}=1$.

(b) Therefore, $\omega_0 N=2\pi m$ for some $m\in\mathbb{I}$ or $\frac{\omega_0}{2\pi}=\frac{m}{N}$.

(c) In other words, $e^{j\omega_0 n}$ is periodic if $\frac{\omega_0}{2\pi}$ is a rational number.

(9) Revisit harmonics of $e^{j\omega_0 n}$ with a common period of N.

(a) $\phi_k[n]=e^{jk(2\pi/N)n}=\phi_{k+N}[n]=e^{j(k+N)(2\pi/N)n}=e^{jk(2\pi/N)n}e^{j2\pi n}$

(b) There are only N distinct periodic exponentials and they are

$\{\phi_0[n], \phi_1[n], \cdots, \phi_{N-1}[n]\}$ or $\left\{1, e^{j2\pi n/N}, e^{j4\pi n/N}, \cdots, e^{j2\pi(N-1)/N}\right\}$ or $\left\{e^{j2\pi k/N}\right\}_{k=0}^{N-1}$.

Continuous-time unit impulse ans unit step function

(1) Unit step function is $u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$. Plot $u(t)$. Is $u(t)$ continuous? Is $u(t)$ differentiable?

(2) Unit impulse function is $\delta(t) = \dfrac{du(t)}{dt}$. And, $u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau = \int_{0}^{\infty} \delta(t-\tau) d\tau$.

(3) Derive $\delta(t)$ and $u(t)$ from $\delta_\Delta(t) = \begin{cases} 1/\Delta, & 0 < t < \Delta \\ 0, & \text{otherwise} \end{cases}$.

(4) Sampling property: $x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$

Discrete-time unit impulse ans unit step function

(1) Unit step is $u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$. Plot $u[n]$.

(2) Unit impulse is $\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$. Plot $\delta[n]$. $\delta[n] = u[n] - u[n-1]$. And,

$$u[n] = \sum_{m=-\infty}^{n} \delta[m] = \sum_{m=0}^{\infty} \delta[n-m].$$

(3) Sampling property: $x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$

System

(1) System transforms input signals into output signals

(2) System is a function mapping input signals into output signals

Input-output representation

Continuous-time system

System transforms input into output: $x(t) \to y(t)$

Discrete-time system

System transforms input into output: $x[n] \to y[n]$

System interconnections

(1) Series or cascade interconnection

(2) Parallel interconnection

(3) Feedback interconnection

(4) Combination

Basic system properties

(1) Memoryless vs. with memory

(2) Invertibility and inverse system

(3) Causality

(4) Stability (boundedness)

(5) Time invariance

(6) Linearity

2. Sampling

Sampling theorem

(1) Impulse train sampling

(a) *Sampling function* is an impulse train,

$$p(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT) \;\Leftrightarrow\; P(j\omega)=\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_s)$$

(b) T is *sampling period* and *sampling frequency* is $\omega_s=\frac{2\pi}{T}$ or $f_s=\frac{\omega_s}{2\pi}=\frac{1}{T}$

(c) $x_p(t)=x(t)p(t)=\sum_{n=-\infty}^{\infty}x(nT)\delta(t-nT) \;\Leftrightarrow$

$$X_p(j\omega)=X(j\omega)*P(j\omega)=\frac{1}{T}\sum_{k=-\infty}^{\infty}X\big(j(\omega-k\omega_s)\big)$$

(2) Sampling theorem:

Let $x(t)$ be a *band-limited* signal with $X(j\omega)=0$ for $|\omega|>\omega_M$. Then $x(t)$ is uniquely determined by its samples $x(nT)$, $n=0,\pm1,\pm2,\cdots$ if $\underline{\omega_s>2\omega_M}$ where $\omega_s=2\pi/T$.

Given these samples, we can reconstruct $x(t)$ by generating a periodic impulse train in which successive impulses have magnitudes that are successive sample values. This impulse train is then processed through an *ideal lowpass filter* with gain T and *cutoff frequency greater than ω_M and less that $\omega_s-\omega_M$*. The resulting output signal will exactly equal $x(t)$.

Signal reconstruction from samples (interpolation)

(1) Reconstructed signal using a filter $h(t)$ is $x_r(t)=x_p(t)*h(t)=\sum_{n=-\infty}^{\infty}x(nT)h(t-nT)$

(2) For an ideal lowpass filter $h(t)$, $h(t)=\frac{\omega_c T\sin(\omega_c t)}{\pi\omega_c t}$ and

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\frac{\omega_c T}{\pi}\frac{\sin\left(\omega_c(t-nT)\right)}{\omega_c(t-nT)}$$

Aliasing: effect of undersampling

C/D and D/C conversion: discrete-time processing of continuous-time signal

(1) Signal conversion

(a) Continuous-to-discrete-time conversion: analog-to-digital converter (A/D converter or ADC)

(b) Discrete-to-continuous-time conversion: digital-to-analog converter (D/A converter or DAC)

(2) Signal representation: $x_d[n] = x_c(nT)$ and $y_d[n] = y_c(nT)$

(3) Note that

$$x_p(t) = x_c(t)p(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t-nT) \Leftrightarrow X_p(j\omega) = \sum_{k=-\infty}^{\infty} x_c(nT)e^{-j\omega nT} \text{ and}$$

$$x_d[n] = x_c(nT) \Leftrightarrow X_d(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x_d[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega n}.$$

Therefore, $X_d(e^{j\Omega}) = X_p(j\Omega/T)$ and $\Omega = \omega T$. Furthermore,

$$X_d(e^{j\Omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j(\Omega - 2\pi k)/T\right)$$

3. Linear Time-Invariant (LTI) Systems

Discrete-time LTI systems: convolution sum

(1) Sampling property: $x[n]=\sum_{k=-\infty}^{\infty}x[k]\delta[n-k]$

(2) Assume $\delta[n-k]\rightarrow h_k[n]$, then $y[n]=\sum_{k=-\infty}^{\infty}x[k]h_k[n]$ from linearity.

(3) $y[n]=\sum_{k=-\infty}^{\infty}x[k]h_k[n]=\sum_{k=-\infty}^{\infty}x[k]h_0[n-k]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]=x[n]*h[n]$ from time-invariance.

Continuous-time LTI systems: convolution integral

(1) Sampling property: $x(t)=\int_{-\infty}^{\infty}x(\tau)\delta(t-\tau)d\tau$

(2) Assume $\delta(t-\tau)\rightarrow h_\tau(t)$, then $y(t)=\int_{-\infty}^{\infty}x(\tau)h_\tau(t)d\tau$ from linearity.

(3) $y(t)=\int_{-\infty}^{\infty}x(\tau)h_\tau(t)d\tau=\int_{-\infty}^{\infty}x(\tau)h_0(t-\tau)d\tau=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau=x(t)*h(t)$ from time-invariance.

Properties of LTI systems

(1) Commutative property: $x[n]*h[n]=h[n]*x[n]$ and $x(t)*h(t)=h(t)*x(t)$.

(2) Distributive property: $x[n]*\left(h_1[n]+h_2[n]\right)=x[n]*h_1[n]+x[n]*h_2[n]$ and

$x(t)*\left\{h_1(t)+h_2(t)\right\}=x(t)*h_1(t)+x(t)*h_2(t)$. Is this parallel interconnection?

(3) Associative property: $x[n]*\left(h_1[n]*h_2[n]\right)=\left(x[n]*h_1[n]\right)*h_2[n]$ and

$x(t)*\left\{h_1(t)*h_2(t)\right\}=\left\{x(t)*h_1(t)\right\}*h_2(t)$. Is this series interconnection?

Memoryless LTI system

$h[n]=\delta[n]$ or $h(t)=\delta(t)$

Invertible LTI system

(1) Discrete-time LTI system is invertible if $\exists h_1[n] \ni h[n]*h_1[n] = \delta[n]$.

(2) Continuous-time LTI system is invertible if $\exists h_1(t) \ni h(t)*h_1(t) = \delta(t)$.

Causal LTI system

(1) Discrete-time LTI system is causal if $h[n]=0$ for $n < 0$. Then, $y[n] = \sum_{k=0}^{\infty} x[k]h[n-k]$.

(2) Continuous-time LTI system is causal if $h(t)=0$ for $t < 0$. Then, $y(t) = \int_0^{\infty} x(\tau)h(t-\tau)d\tau$.

Stable LTI system

(1) Discrete-time LTI system is causal if $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$, i.e. absolutely summable.

(2) Continuous-time LTI system is causal if $\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$, i.e. absolutely integrable.

Unit step response of LTI system

(1) For discrete-time LTI system, $s[n] = u[n]*h[n] = \sum_{k=-\infty}^{\infty} h[k]$, $h[n] = s[n] - s[n-1]$.

(2) For continuous-time LTI system, $s(t) = u(t)*h(t) = \int_{-\infty}^{t} h(\tau)d\tau$, $h(t) = \frac{ds(t)}{dt}$.

Linear constant-coefficient differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \Leftrightarrow x(t) \xrightarrow{\text{Causal LTI}} y(t)$$

Linear constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \Leftrightarrow x[n] \xrightarrow{\text{Causal LTI}} y[n]$$

4. Fourier Series

Response of LTI system to complex exponentials

(1) Eigenfunctions and eigenvalues

(a) Continuous-time ($e^{st} \to H(s)e^{st}$): if $x(t)=e^{st}$,

$$y(t)=\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau=\int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau=e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

Define the <u>system function</u>, $H(s)=\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$. Then, $y(t)=H(s)e^{st}$.

(b) Discrete-time ($z^n \to H(z)z^n$): if $x[n]=z^n$,

$$y[n]=\sum_{k=-\infty}^{\infty} h[k]x[n-k]=\sum_{k=-\infty}^{\infty} h[k]z^{n-k}=z^n\sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

Define the <u>system function</u>, $H(z)=\sum_{k=-\infty}^{\infty} h[k]z^{-k}$. Then, $y[n]=H(z)z^n$.

(2) Superposition principle

(a) Continuous-time: if $x(t)=\sum_k a_k e^{s_k t}$, $y(t)=\sum_k a_k H(s_k)e^{s_k t}$.

(a) Discrete-time: if $x[n]=\sum_k a_k z_k^n$, $y[n]=\sum_k a_k H(z_k)z_k^n$.

Fourier series representation of continuous-time periodic signal

(1) Representation

$$x(t)=x(t+T) \quad \text{for all } t$$

$$\Downarrow$$

$$x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}=\sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}=a_0+\sum_{k=1}^{\infty}\left[a_k e^{jk\omega_0 t}+a_{-k}e^{-jk\omega_0 t}\right]$$

If signal is real, $x(t)=x^*(t)$. Then, $a_k^*=a_{-k}$ and

$$x(t)=a_0+\sum_{k=1}^{\infty}\left[a_k e^{jk\omega_0 t}+a_k^* e^{-jk\omega_0 t}\right]=a_0+2\sum_{k=1}^{\infty} A_k\cos\left(k\omega_0 t+\theta_k\right) \quad \text{with} \quad a_k=A_k e^{j\theta_k}$$

or

$$x(t)=a_0+2\sum_{k=1}^{\infty}\left[B_k\cos k\omega_0 t-C_k\sin k\omega_0 t\right] \quad \text{with} \quad a_k=B_k+jC_k.$$

(2) Coefficient determination

$$x(t)e^{-jn\omega_0 t}=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}e^{-jn\omega_0 t}$$

$$\int_0^T x(t)e^{-jn\omega_0 t}dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}dt = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t}dt$$

$$\int_0^T e^{j(k-n)\omega_0 t}dt = \int_0^T \cos(k-n)\omega_0 t dt + j\int_0^T \sin(k-n)\omega_0 t dt = \begin{cases} T, & \text{if } k=n \\ 0, & \text{if } k \neq n \end{cases}$$

$$a_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jn\omega_0 t}dt$$

(3) Fourier series pair of periodic continuous-time signal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$$

where $\{a_k\}$ are Fourier series coefficients or spectral coefficients of $x(t)$. Note that $a_0 = \frac{1}{T}\int_T x(t)dt = \text{average value over one period}$.

(4) Convergence of the Fourier series

(a) Over any period, $x(t)$ must be absolutely integrable; that is,

$$\int_T |x(t)| dt < \infty \quad \text{so that} \quad |a_k| < \infty .$$

(b) In any finite interval of time, $x(t)$ is of bounded variation; that is, there are no more than a finite number of maxima and minima during any single period of the signal.

(c) In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

Properties of continuous-time Fourier series

- Both $x(t)$ and $y(t)$ are periodic with T and $\omega_0 = 2\pi/T$. Their Fourier coefficients are a_k and b_k, respectively.

	Signal	Fourier coefficient
Linearity	$Ax(t)+By(t)$	$Aa_k + Bb_k$
Time shifting	$x(t-t_0)$	$a_k e^{-jk\omega_0 t_0}$
Frequency shifting	$e^{jM\omega_0 t}x(t)$	a_{k-M}
Conjugation	$x^*(t)$	a_{-k}^*
Time reversal	$x(-t)$	a_{-k}
Time scaling	$x(\alpha t),\ \alpha > 0$, period of T/α	a_k
Periodic convolution	$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$

Multiplication	$x(t)y(t)$	$\sum_{l=-\infty}^{\infty} a_l b_{k-l}$
Differentiation	$\frac{dx(t)}{dt}$	$jk\omega_0 a_k$
Integration	$\int_{-\infty}^{t} x(t)dt$, finite with $a_0 = 0$	$\left(\frac{1}{jk\omega_0}\right) a_k$
Conjugate symmetry	Real $x(t)$	$a_k = a_{-k}^*$ $(a_k(= (a_{-k}($ and $\angle a_k = -\angle a_{-k}$ $\mathrm{Re}\{a_k\} = \mathrm{Re}\{a_{-k}\}$ and $\mathrm{Im}\{a_k\} = -\mathrm{Im}\{a_{-k}\}$
Real and even	Real and even $x(t)$	a_k real and even
Real and odd	Real and odd $x(t)$	a_k purely imaginary and odd
Even decomposition	$x_e(t) = \frac{1}{2}\{x(t) + x(-t)\}$, real	$\mathrm{Re}\{a_k\}$
Odd decomposition	$x_o(t) = \frac{1}{2}\{x(t) - x(-t)\}$, real	$j\,\mathrm{Im}\{a_k\}$
Parseval's relation	$\frac{1}{T}\int_T (x(t)(^2 dt = \sum_{k=-\infty}^{\infty} (a_k(^2$	

Fourier series representation of discrete-time periodic signal

(1) Representation

$$x[n] = x[n+N] \quad \text{for all } n$$

$$\Downarrow$$

$$x[n] = \sum_{k=(N(} a_k \phi_k[n] = \sum_{k=(N(} a_k e^{jk\omega_0 n} = \sum_{k=(N(} a_k ejk(2\pi/N)n$$

(2) Coefficient determination

$$a_k = \frac{1}{N}\sum_{n=(N(} x[n]e^{-jk(2\pi/N)n}$$

(3) Fourier series pair of periodic continuous-time signal

$$x[n] = \sum_{k=(N(} a_k e^{jk\omega_0 n} = \sum_{k=(N(} a_k ejk(2\pi/N)n$$

$$a_k = \frac{1}{N}\sum_{n=(N(} x[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=(N(} x[n]e^{-jk(2\pi/N)n}$$

where $\{a_k\}$ are spectral coefficients of $x[n]$.

Properties of discrete-time Fourier series

- Both $x[n]$ and $y[n]$ are periodic with N and $\omega_0 = 2\pi/N$. Their Fourier series coefficients are a_k and b_k , respectively, and are periodic with N.

	Signal	Fourier coefficient
Linearity	$Ax[n]+By[n]$	$Aa_k + Bb_k$
Time shifting	$x[n-n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency shifting	$e^{jM(2\pi/N)t}x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time reversal	$x[-n]$	a_{-k}
Time scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n = lm \\ 0, & \text{otherwise} \end{cases}$ where l is an integer	$\frac{1}{m}a_k$
Periodic convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-l}$
First difference	$x[n]-x[n-1]$	$\left(1-e^{-jk(2\pi/N)}\right)a_k$
Running sum	$\sum_{k=-\infty}^{n} x[k]$, finite and periodic with $a_0 = 0$	$\left(\frac{1}{1-e^{-jk(2\pi/N)}}\right)a_k$
Conjugate symmetry	Real $x[n]$	$a_k = a^*_{-k}$ $\lvert a_k\rvert = \lvert a_{-k}\rvert$ and $\angle a_k = -\angle a_{-k}$ $\mathrm{Re}\{a_k\} = \mathrm{Re}\{a_{-k}\}$ and $\mathrm{Im}\{a_k\} = -\mathrm{Im}\{a_{-k}\}$
Real and even	Real and even $x[n]$	a_k real and even
Real and odd	Real and odd $x[n]$	a_k purely imaginary and odd
Even decomposition	$x_e[n] = \frac{1}{2}\{x[n]+x[-n]\}$, real	$\mathrm{Re}\{a_k\}$
Odd decomposition	$x_o[n] = \frac{1}{2}\{x[n]-x[-n]\}$, real	$j\,\mathrm{Im}\{a_k\}$
Parseval's relation	$\frac{1}{N}\sum_{n=\langle N\rangle} \lvert x[n]\rvert^2 = \sum_{k=\langle N\rangle} \lvert a_k\rvert^2$	

Fourier series and LTI systems

(1) Frequency response

(a) Continuous-time: $H(j\omega) = H(s)\big|_{s=j\omega} = \int_{-\infty}^{\infty} h(t)e^{j\omega t}dt$

(b) Discrete-time: $H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$

(2) Superposition principle

(a) Continuous-time: if $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$, $y(t) = \sum_{k=-\infty}^{\infty} a_k H(e^{jk\omega_0}) e^{jk\omega_0 t}$.

(a) Discrete-time:

if $x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$, $y[n] = \sum_{k=\langle N \rangle} a_k H(e^{j2\pi k/N}) e^{jk(2\pi/N)n}$.

Filtering

(1) Frequency-shaping filters

(2) Frequency-selective filters

5. Continuous-Time Fourier Transform

Fourier transform pair

(1) Fourier transform or Fourier integral

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

(2) Inverse Fourier transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

where $X(j\omega)$ is called the spectrum of $x(t)$.

(3) For a periodic signal $\tilde{x}(t)$ with period T,

$$a_k = \frac{1}{T}X(j\omega)\Big|_{\omega=k\omega_0} \quad \text{since}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2}\tilde{x}(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-\infty}^{\infty}x(t)e^{-jk\omega_0 t}dt \quad \text{with} \quad x(t) = \begin{cases}\tilde{x}(t), & -T/2 \le t \le T/2 \\ 0, & \text{otherwise}\end{cases}.$$

(4) Convergence of the Fourier transform

(a) $x(t)$ is absolutely integrable; that is,

$$\int_{-\infty}^{\infty}|x(t)|dt < \infty .$$

(b) $x(t)$ has a finite number of maxima and minima within any finite interval.

(c) $x(t)$ has a finite number of discontinuities within any finite interval. Furthermore, each of these discontinuities must be finite.

Fourier transform for periodic signals

(1) $X(j\omega) = 2\pi\delta(\omega-\omega_0) \Leftrightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}2\pi\delta(\omega-\omega_0)e^{j\omega t}d\omega = e^{j\omega_0 t}$

(2) In general, $X(j\omega) = \sum_{k=-\infty}^{\infty}2\pi a_k\delta(\omega-k\omega_0) \Leftrightarrow x(t) = \sum_{k=-\infty}^{\infty}a_k e^{jk\omega_0 t}$

Properties of continuous-time Fourier transform

- $x(t) \Leftrightarrow X(j\omega)$ and $y(t) \Leftrightarrow Y(j\omega)$

	Signal	Fourier transform
Linearity	$ax(t)+by(t)$	$aX(j\omega)+bY(j\omega)$
Time shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(j\omega)$
Frequency shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time reversal	$x(-t)$	$X(-j\omega)$
Time/frequency scaling	$x(\alpha t)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
Convolution	$x(t)*y(t)$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}X(j\omega)*Y(j\omega)$
Differentiation	$\frac{dx(t)}{dt}$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t}x(t)dt$	$\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$
Conjugate symmetry	Real $x(t)$	$X(j\omega)=X^*(-j\omega)$ $\lvert X(j\omega)\rvert=\lvert X(-j\omega)\rvert$ $\angle X(j\omega)=-\angle X(-j\omega)$ $\mathrm{Re}\{X(j\omega)\}=\mathrm{Re}\{X(-j\omega)\}$ $\mathrm{Im}\{X(j\omega)\}=-\mathrm{Im}\{X(-j\omega)\}$
Real and even	Real and even $x(t)$	$X(j\omega)$ real and even
Real and odd	Real and odd $x(t)$	$X(j\omega)$ purely imaginary and odd
Even decomposition	$x_e(t)=\frac{1}{2}\{x(t)+x(-t)\}$, real	$\mathrm{Re}\{X(j\omega)\}$
Odd decomposition	$x_o(t)=\frac{1}{2}\{x(t)-x(-t)\}$, real	$j\,\mathrm{Im}\{X(j\omega)\}$
Parseval's relation	$\int_{-\infty}^{\infty}\lvert x(t)\rvert^2 dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(j\omega)\rvert^2 d\omega$	

System characterized by linear constant-coefficient differential equation

(1) A class of continuous-time LTI system with

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \Leftrightarrow x(t) \xrightarrow{\text{Causal LTI}} y(t)$$

(2) Frequency response

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

6. Laplace Transform

(Bilateral) Laplace transform

(1) Definition with $s = \sigma + j\omega$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \iff x(t) \xleftrightarrow{\mathcal{L}} X(s)$$

(2) Fourier transform

$$\mathcal{F}\{x(t)\} = X(j\omega) = X(s)|_{s=j\omega}$$

(3) ROC (region of convergence)

(4) Calculus problem ⇔ algebraic problem

Rational Laplace transform

(1) $X(s) = \dfrac{N(s)}{D(s)}$

(2) Poles

(3) Zeros

ROC

(1) The ROC of $X(s)$ consists of strips parallel to the $j\omega$-axis in the s-plane.

(2) For rational Laplace transforms, the ROC does not contain any poles.

(3) If $x(t)$ is of finite duration and is absolutely integrable, then the ROC is the entire s-plane.

(4) If $x(t)$ is right-sided, and if the line $\text{Re}\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\text{Re}\{s\} > \sigma_0$ will also be in the ROC.

(5) If $x(t)$ is left-sided, and if the line $\text{Re}\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\text{Re}\{s\} < \sigma_0$ will also be in the ROC.

(6) If $x(t)$ is two-sided, and if the line $\text{Re}\{s\} = \sigma_0$ is in the ROC, then the ROC will consist of a strip in the s-plane that includes the line $\text{Re}\{s\} = \sigma_0$.

(7) If the Laplace transform $X(s)$ of $x(t)$ is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles of $X(s)$ are contained in the ROC.

(8) If the Laplace transform $X(s)$ of $x(t)$ is rational, then if $x(t)$ is right-sided, the ROC is the region in the s-plane to the right of the rightmost pole. If $x(t)$ is left-sided, the ROC is the region in the s-plane to the left of the leftmost pole.

Inverse Laplace transform

(1) Definition with $s = \sigma + j\omega$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds \Leftrightarrow x(t) \overset{L}{\longleftrightarrow} X(s)$$

(2) In practice, for rational Laplace transforms, use the partial fration expansion.

Geometric evaluation of frequency response from pole-zero plot

Properties of Laplace transform

	Signal	*Laplace transform*	*ROC*
	$x(t)$ $x_1(t)$ $x_2(t)$	$X(s)$ $X_1(s)$ $X_2(s)$	R R_1 R_2
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
Shifting in the s-domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R, i.e. $s \in$ROC if $(s - s_0) \in R$
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	Scaled ROC, i.e., $s \in$ROC if $s/a \in R$
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Differentiation in the time domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
Differentiation in the *s*-domain	$-t\,x(t)$	$\frac{d}{ds}X(s)$	R
Integration in the time domain	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	At least $R \cap \{\text{Re}\{s\} > 0\}$
Initial- and final-value problem	$x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher order singularities at $t = 0$	$x(0^+) = \lim_{s\to\infty} sX(s)$ $\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$	

LTI system and Laplace transform

(1) $Y(s) = H(s)X(s)$ where $H(s)$ is the transfer function or system function

(2) Causility

(a) The ROC of $H(s)$ for causal LTI system is a right-half plane

(b) For a rational $H(s)$, the ROC of $H(s)$ for causal LTI system is the right-half plane to the right of the rightmost pole

(3) Stability

(a) An LTI system is stable iff the ROC of its $H(s)$ includes the entire $j\omega$-axis

(b) A causal LTI system with rational $H(s)$ is stable iff all poles of $H(s)$ lie in the left-half plane, i.e., all poles have negative real parts

System function algebra and block diagram representations

(1) Parallel interconnection: $h(t) = h_1(t) + h_2(t) \Leftrightarrow H(s) = H_1(s) + H_2(s)$

(2) Series or cascade interconnection: $h(t) = h_1(t) * h_2(t) \Leftrightarrow H(s) = H_1(s)H_2(s)$

(3) Feedback interconnection: $H(s) = \dfrac{Y(s)}{X(s)} = \dfrac{H_1(s)}{1 + H_1(s)H_2(s)}$

Unilateral Laplace transform

(a) Definition: $\mathcal{X}(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt \Leftrightarrow x(t) \xleftrightarrow{\mathcal{UL}} \mathcal{X}(s) = \mathcal{UL}\{x(t)\}$

(b) ROC is always a right-half plane

7. Discrete-Time Fourier Transform

Discrete-time Fourier transform pair

(1) Discrete-time Fourier transform

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

(2) Inverse Fourier transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

where $X(e^{j\omega})$ is called the spectrum of $x[n]$.

(3) Convergence of the discrete-time Fourier transform: $x[n]$ is absolutely summable; that is,

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty \quad \text{or} \quad \sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty .$$

Discrete-time Fourier transform for periodic signals

(1) $x[n] = e^{j\omega_0 n} \Leftrightarrow X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$

(2) In general, $x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n} \Leftrightarrow X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - \frac{2\pi k}{N})$

Properties of discrete-time Fourier transform

- $x[n] \Leftrightarrow X(e^{j\omega})$ and $y[n] \Leftrightarrow Y(e^{j\omega})$

	Signal	Fourier transform
Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
Time shifting	$x[n - n_0]$	$e^{-j\omega n_0} X(e^{j\omega})$
Frequency shifting	$e^{j\omega_0 n} x[n]$	$X\left(e^{j(\omega-\omega_0)}\right)$
Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
Time reversal	$x[-n]$	$X(e^{-j\omega})$

Time expansion	$x_k[n]=\begin{cases} x[n/k], & \text{if } n=mk \\ 0, & \text{otherwise}\end{cases}$ where m is an integer	$X\left(e^{jk\omega}\right)$
Convolution	$x[n]*y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi}X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Differencing in time	$x[n]-x[n-1]$	$(1-e^{-j\omega})X(e^{j\omega})$
Accumulation	$\sum_{k=-\infty}^{n}x[k]$	$\frac{1}{1-e^{-j\omega}}X(e^{j\omega})+$ $\pi X(e^{j0})\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)$
Differentiation in frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
Conjugate symmetry	Real $x[n]$	$X(e^{j\omega})=X^*(e^{-j\omega})$ $\left\|X(e^{j\omega})\right\|=\left\|X(e^{-j\omega})\right\|$ $\angle X(e^{j\omega})=-\angle X(e^{-j\omega})$ $\mathrm{Re}\left\{X(e^{j\omega})\right\}=\mathrm{Re}\left\{X(e^{-j\omega})\right\}$ $\mathrm{Im}\left\{X(e^{j\omega})\right\}=-\mathrm{Im}\left\{X(e^{-j\omega})\right\}$
Real and even	Real and even $x[n]$	$X(e^{j\omega})$ real and even
Real and odd	Real and odd $x[n]$	$X(e^{j\omega})$ purely imaginary and odd
Even decomposition	$x_e[n]=\frac{1}{2}\{x[n]+x[-n]\}$, real	$\mathrm{Re}\left\{X(e^{j\omega})\right\}$
Odd decomposition	$x_o[n]=\frac{1}{2}\{x[n]-x[-n]\}$, real	$j\,\mathrm{Im}\left\{X(e^{j\omega})\right\}$
Parseval's relation	$\sum_{n=-\infty}^{\infty}\left\|x[n]\right\|^2=\frac{1}{2\pi}\int_{2\pi}\left\|X(e^{j\omega})\right\|^2 d\omega$	

System characterized by linear constant-coefficient difference equation

(1) A class of discrete-time LTI system with

$$\sum_{k=0}^{N}a_k y[n-k]=\sum_{k=0}^{M}b_k x[n-k]\Leftrightarrow x[n]\xrightarrow{\text{Causal LTI}}y[n]$$

(2) Frequency response

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}$$

8. Z-Transform

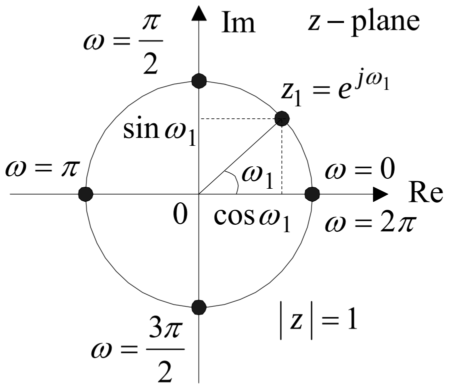
(Bilateral) z-transform

(1) 정의 $(z = re^{j\omega})$

$$\mathcal{Z}\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \quad \overset{Z}{\longleftrightarrow} \quad x[n] \overset{Z}{\longleftrightarrow} X(z)$$

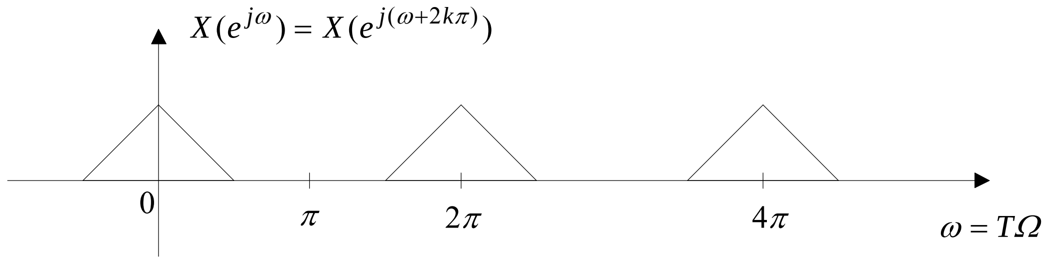
(2) Fourier transform 과의 관계

$$\mathcal{F}\{x[n]\} = X(e^{j\omega}) = X(z)\big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad \overset{Z}{\longleftrightarrow} \quad x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$



$$z = re^{j\omega} = r(\cos\omega + j\sin\omega) \Leftrightarrow |z| = r \ge 0 \text{ and } \angle z = \omega$$

$$e^{j\omega} = \cos\omega + j\sin\omega \text{ and } |e^{j\omega}| = \sqrt{\cos^2\omega + \sin^2\omega} = 1$$



(3) ROC (region of convergence)

(a) 수렴 조건: $\sum_{n=-\infty}^{\infty} |x[n]||z|^{-n} < \infty \Rightarrow |X(z)| < \infty$

(b) $z = z_1 \in \text{ROC} \;\Rightarrow\; \{z : |z| = |z_1|\} \subset \text{ROC}$: ROC 는 원점을 중심으로 하는 ring 의 모양을 가진다.

(c) $\{z : |z| = 1\} \subset \text{ROC} \;\Rightarrow\; X(e^{j\omega})$ 가 수렴, $X(e^{j\omega})$ 가 analytic 한 경우;

$X(e^{j\omega}) = \delta(\omega - \omega_0) + \delta(\omega + \omega_0)$ 인 경우는?

Rational form

(1) $X(z) = \dfrac{N(z)}{D(z)}$

(a) $x[n]$이 real 또는 complex exponentials 의 합으로 표현될 때

(b) $x[n]$의 길이가 유한할 때

(2) Pole 과 zero

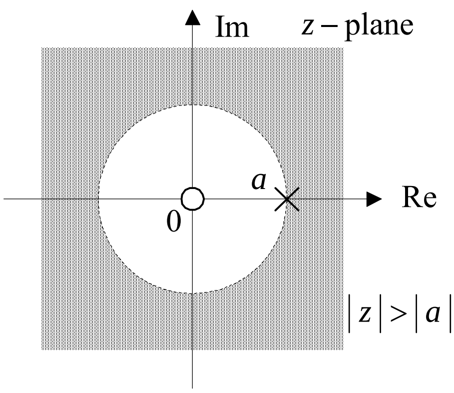
(a) $X(z_1) = 0 \;\Rightarrow\; z = z_1$을 zero 라 함

(b) $X(z_1) \to \infty \;\Rightarrow\; z = z_1$을 pole 이라 함

(3) 예제: $x[n] = a^n u[n]$

$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n$ 이고, $X(z)$ 가 수렴하기 위해서는 $\left|az^{-1}\right| < 1$

즉, $|z| > |a|$ 이어야 함. 따라서, $X(z) = \dfrac{1}{1 - az^{-1}} = \dfrac{z}{z - a}$ 이고 ROC 는 $|z| > |a|$이다.



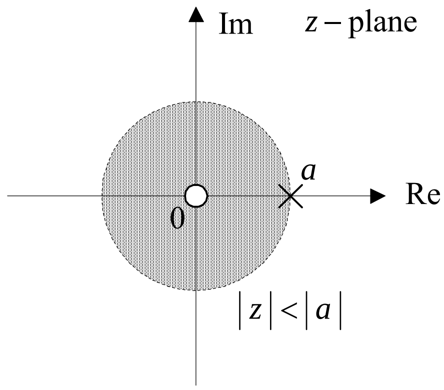
만약, $a = 1$ 이라면, $X(z) = \dfrac{1}{1 - z^{-1}} = \dfrac{z}{z - 1}$ $(|z| > 1)$이다.

(4) 예제: $x[n] = -a^n u[-n-1]$

$$X(z) = \sum_{n=-\infty}^{\infty} \left(-a^n u[-n-1]\right) z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=1}^{\infty} \left(a^{-1}z\right)^n = -\frac{a^{-1}z}{1-a^{-1}z} = \frac{z}{z-a}$$ 이

고, $X(z)$ 가 수렴하기 위해서는 $\left|a^{-1}z\right| < 1$ 즉, $|z| < |a|$ 이어야 함. 따라서,

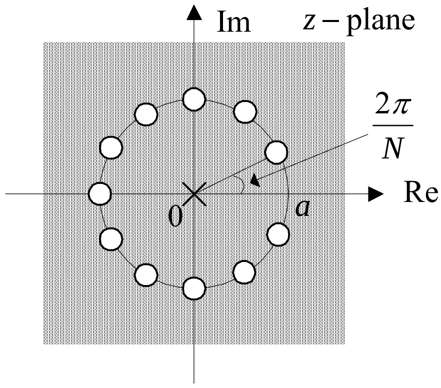
$X(z) = \dfrac{z}{z-a}$ 이고 ROC 는 $|z| < |a|$ 이다.



만약, $a = 1$ 이라면, $X(z) = \dfrac{z}{z-1}$ $(|z| < 1)$이다.

(5) 예제: $x[n] = \begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} \left(az^{-1}\right)^n = \frac{1-\left(az^{-1}\right)^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$ 이고, $z = 0$ 에 $(N-1)$개의 pole 이 있고, $(N-1)$개의 zero 가 $z_k = ae^{j2\pi k/N}$, $k = 1, 2, \cdots, N-1$ 에 있다. 따라서, ROC 는 $z \ne 0$ 인 전체 공간이 된다.



선형성 (Linearity)

(1) $\mathcal{Z}\left\{ax_1[n]+bx_2[n]\right\}=aX_1(z)+bX_2(z)$

(2) $x[n]=\sum_{i=1}^{K}a_i^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)=\sum_{i=1}^{K}\frac{1}{1-a_i z^{-1}}=\sum_{i=1}^{K}\frac{z}{z-a_i},\ \left\{z : |z| > \max_i |a_i|\right\}$

(3) $x[n]=\sum_{i=1}^{K}b_i^n u[-n-] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)=\sum_{i=1}^{K}\frac{1}{1-b_i z^{-1}}=\sum_{i=1}^{K}\frac{z}{z-b_i},\ \left\{z : |z| > \min_i |b_i|\right\}$

(4) 예제: $x[n]=\left(\frac{1}{2}\right)^n u[n]+\left(-\frac{1}{3}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow}\ ?$

(5) 예제: $x[n]=\left(\frac{1}{4}\right)^n u[n]-\left(\frac{1}{2}\right)^n u[-n-1] \overset{\mathcal{Z}}{\longleftrightarrow}\ ?$

ROC 의 성질

(1) The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin.

(2) The ROC does not contain any poles.

(3) If $x[n]$ is of finite duration, then the ROC is the entire z-plane except possible $z = 0$ and/or $z = \infty$.

(4) If $x[n]$ is a right-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $|z| > r_0$ will also be in the ROC.

(5) If $x[n]$ is a left-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all finite

values of z for which $0 < |z| < r_0$ will also be in the ROC.

(6) If $x[n]$ is two-sided, and if the circle $|z| = r_0$ is in the ROC, then the ROC will consist of a ring in the z-plane that includes the circle $|z| = r_0$.

(7) If the z-transform $X(z)$ of $x[n]$ is rational, then its ROC is bounded by poles or extends to infinity.

(8) If the z-transform $X(z)$ of $x[n]$ is rational, then if $x[n]$ is right-sided, the ROC is the region in the z-plane outside the outermost pole. If $x[n]$ is causal, the ROC also includes $z = \infty$.

Inverse z-transform

(1) Definition with $z = re^{j\omega}$ (실제로는 거의 사용하지 않음)

$$x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz \stackrel{Z}{\longleftrightarrow} x[n] \stackrel{Z}{\longleftrightarrow} X(z)$$

(2) In practice, for rational z-transforms, use inspection or partial fration expansion.

(a) Inspection

$$a^n u[n] \stackrel{Z}{\longleftrightarrow} \frac{1}{1-az^{-1}}, |z| > |a|$$

$$a^n u[-n-] \stackrel{Z}{\longleftrightarrow} \frac{1}{1-az^{-1}}, |z| < |a|$$

(b) Partial fraction expansion

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{z^N \sum_{k=0}^{M} b_k z^{M-k}}{z^M \sum_{k=0}^{N} a_k z^{N-k}}$$

- $M > N \Rightarrow M - N$ poles at $z = 0$
- $M < N \Rightarrow N - M$ zeros at $z = 0$
- M zeros at $z \neq 0$ and N poles at $z \neq 0$

$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} \left(1 - c_k z^{-k}\right)}{\prod_{k=1}^{N} \left(1 - d_k z^{-k}\right)}$$

$$X(z) = \underbrace{\sum_{r=0}^{M-N} B_r z^{-r}}_{\text{if } M>N} + \underbrace{\sum_{k=1,k\neq i}^{N} \frac{A_k}{1-d_k z^{-1}}}_{\text{single poles}} + \underbrace{\sum_{m=1}^{s} \frac{C_m}{\left(1-d_i z^{-1}\right)^m}}_{\text{multiple pole of order } m}$$

$$A_k = \left(1-d_k z^{-1}\right) X(z)\big|_{z=d_k}$$

$$C_m = \frac{1}{(s-m)!(-d_i)^{s-m}} \left\{ \frac{d^{s-m}}{dw^{s-m}} \left[(1-d_i w)^s X\left(w^{-1}\right) \right] \right\}_{w=d_i^{-1}}$$

$$B_r z^{-r} \Leftrightarrow B_r \delta[n-r]$$

$$\frac{A_k}{1-d_k z^{-1}} \Leftrightarrow \begin{cases} A_k (d_k)^n u[n], & |z| > d_k \\ A_k (d_k)^n u[-n-1], & |z| < d_k \end{cases}$$

예제: $X(z) = \dfrac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}} = 2 - \dfrac{9}{1-\frac{1}{2}z^{-1}} + \dfrac{8}{1-z^{-1}}$ 및 $|z|>1$

(c) Power series expansion: $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

예제:

$$X(z) = z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1} \xleftrightarrow{Z} x[n] = \delta[n+2] - \frac{1}{2}\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1]$$

예제:

$$X(z) = \log\left(1+az^{-1}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}, \quad |z|>|a| \quad \xleftrightarrow{Z} \quad x[n] = \begin{cases} (-1)^{n+1} a^n, & n>1 \\ 0, & n \le 0 \end{cases}$$

Geometric evaluation of frequency response from pole-zero plot

Properties of z-transform

	Signal	*Laplace transform*	*ROC*
	$x[n]$ $x_1[n]$ $x_2[n]$	$X(z)$ $X_1(z)$ $X_2(z)$	R R_1 R_2
Linearity	$ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$	At least $R_1 \cap R_2$
Time shifting	$x[n-n_0]$	$z^{-n_0}X(z)$	R, except possible addition or deletion of the origin
Scaling in the z-domain	$e^{j\omega_0 n}x[n]$ $z_0^n x[n]$ $a^n x[n]$	$X(e^{-j\omega_0}z)$ $X(z/z_0)$ $X(a^{-1}z)$	R $z_0 R$ Scaled version of R
Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R
Time expansion	$x_{(k)}[n]=\begin{cases} x[r], & n=rk \\ 0, & n \neq rk \end{cases}$	$X(z^k)$	$R^{1/k}$
Conjugation	$x^*[n]$	$X^*(z^*)$	R
Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least $R_1 \cap R_2$
First difference	$x[n]-x[n-1]$	$(1-z^{-1})X(z)$	At least $R \cap \{\lvert z \rvert > 0\}$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1-z^{-1}}X(z)$	At least $R \cap \{\lvert z \rvert > 1\}$
Differentiation in the z-domain	$n\,x[n]$	$-z\frac{dX(z)}{dz}$	R
Initial-value problem	$x[n]=0$ for $n<0$	$x[0]=\lim_{z\to\infty} X(z)$	

예제: $$x[n]=r^n\cos(\omega_0 n)u[n]=\frac{1}{2}r^n e^{j\omega_0 n}u[n]+\frac{1}{2}r^n e^{-j\omega_0 n}u[n]$$

$$u[n] \overset{Z}{\longleftrightarrow} \frac{1}{1-z^{-1}}, \quad |z|>1$$

$$\frac{1}{2}r^n e^{j\omega_0 n}u[n] \overset{Z}{\longleftrightarrow} \frac{1/2}{1-\left(\frac{z}{re^{j\omega_0}}\right)^{-1}}=\frac{1/2}{1-re^{j\omega_0}z^{-1}}$$

$$\frac{1}{2} r^n e^{-j\omega_0 n} u[n] \overset{Z}{\longleftrightarrow} \frac{1/2}{1-\left(\frac{z}{re^{-j\omega_0}}\right)^{-1}} = \frac{1/2}{1-re^{-j\omega_0} z^{-1}}$$

$$X(z) = \frac{1 - r\cos\omega_0 z^{-1}}{1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2}}, \quad |z| > r$$

Complex convolution

(1) $w[n] = x_1[n]x_2[n] \overset{Z}{\longleftrightarrow} W(z) = \frac{1}{2\pi j}\oint_C X_1\left(\frac{z}{v}\right) X_2(v) v^{-1} dv$

(2) $w[n] = x_1[n]x_2[n] \overset{F}{\longleftrightarrow} W(z) = \frac{1}{2\pi} X_1\left(e^{j\omega}\right) * X_2\left(e^{j\omega}\right)$, periodic convolution

Parseval's relation

(1) $\sum_{n=-\infty}^{\infty} x_1[n] x_2^*[n] = \frac{1}{2\pi j}\oint_C X_1(v) X_2\left(1/v^*\right) v^{-1} dv$

(2) $\sum_{n=-\infty}^{\infty} x_1[n] x_2^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1\left(e^{j\omega}\right) X_2^*\left(e^{j\omega}\right) d\omega$

Autocorrelation

(1) 정의: $C_{xx}[n] = \sum_{k=-\infty}^{\infty} x[k]x[n+k]$

(2) z-transform:

$$\begin{aligned} C_{xx}(z) &= \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x[k]x[n+k]z^{-n} = \sum_{k=-\infty}^{\infty} x[k] \sum_{n=-\infty}^{\infty} x[n+k]z^{-n} \\ &= \sum_{k=-\infty}^{\infty} x[k] z^k X(z) = X(z)X\left(z^{-1}\right) \end{aligned}$$

LTI system and z transform

(1) $Y(z) = H(z)X(z)$ where $H(z)$ is the transfer function or system function

(2) Causility

(a) A discrete-time LTI system is causal iff the ROC of $H(z)$ is the exterior of a circle, including infinity

(b) A discrete-time LTI system with rational $H(z)$ is causal iff

- the ROC is the exterior of a circle outside the outermost pole, and
- the order of the numerator of $H(z)$ cannot be greater that the order of the denominator

(3) Stability

(a) An LTI system is stable iff the ROC of its $H(s)$ includes the unit circle

(b) A causal LTI system with rational $H(s)$ is stable iff all poles of $H(s)$ lie inside the unit circle, i.e., all poles have magnitudes smaller than one

System function algebra and block diagram representations

(1) Parallel interconnection: $h[n]=h_1[n]+h_2[n] \Leftrightarrow H(z)=H_1(z)+H_2(z)$

(2) Series or cascade interconnection: $h[n]=h_1[n]*h_2[n] \Leftrightarrow H(z)=H_1(z)H_2(z)$

(3) Feedback interconnection: $H(z)=\dfrac{Y(z)}{X(z)}=\dfrac{H_1(z)}{1+H_1(z)H_2(z)}$

Unilateral z-transform

(1) Definition: $\mathcal{X}(z)=\sum_{n=0}^{\infty} x[n]z^{-n} \Leftrightarrow x[n] \xleftrightarrow{\mathcal{UZ}} \mathcal{X}(z)=\mathcal{UZ}\{x[n]\}$

(2) ROC is always the exterior of a circle

9. Time and Frequency Characterization of Signals and Systems

Magnitude-phase representation of Fourier transform

(1) $X(j\omega)=|X(j\omega)|e^{j\angle X(j\omega)}$ or $X(e^{j\omega})=\left|X(e^{j\omega})\right|e^{j\angle X(e^{j\omega})}$

(2) Magnitude

(a) $|X(j\omega)|^2$: energy density spectrum

(b) $|X(j\omega)|^2\dfrac{d\omega}{2\pi}$: energy in the frequency band $[\omega,\omega+d\omega]$

(c) $|X(j\omega)|$: relative magnitudes of complex exponentials that make up $x(t)$

(3) Phase

(a) $\angle|X(j\omega)|$: relative phase of complex exponentials that make up $x(t)$ and greatly affects the signal

(b) Important in some case (image) and not important in some other case (sound)

LTI system

(1) $Y(j\omega)=H(j\omega)X(j\omega)$ or $Y(e^{j\omega})=H(e^{j\omega})X(e^{j\omega})$

(2) $|Y(j\omega)|=|H(j\omega)||X(j\omega)|$ where $|H(j\omega)|$ is the gain or magnitude distortion

(3) $\angle Y(j\omega)=\angle H(j\omega)+\angle X(j\omega)$ where $\angle H(j\omega)$ is the phase shift or phase distortion

(4) Linear and nonlinear phase

(a) Linear phase: delay without distortion,

$$H(j\omega)=e^{-j\omega t_0}\Leftrightarrow y(t)=x(t-t_0)\quad\Leftrightarrow \text{delay of } t_0$$

$$H(e^{j\omega})=e^{-j\omega n_0}\Leftrightarrow y[n]=x[n-n_0]\quad\Leftrightarrow \text{delay of } n_0$$

(b) Nonlinear phase: delay with distortion

(5) All-pass system: $|H(j\omega)|=1$ or $\left|H(e^{j\omega})\right|=1$

(6) Group delay: $\tau(\omega)=-\dfrac{d}{d\omega}\{\angle H(j\omega)\}$ or $\tau(\omega)=-\dfrac{d}{d\omega}\left\{\angle H(e^{j\omega})\right\}$

(a) Principal phase
(b) Unwrapped phase
(c) Dispersion: different frequency components delayed by different amounts

Bode plot

(1) Continuous-time

(a) Magnitude: $20\log_{10}|H(j\omega)|$ versus $\log_{10} f$

(b) Phase: $\angle H(j\omega)$ versus $\log_{10} f$

(2) Discrete-time

(a) Magnitude: $20\log_{10}\left|H(e^{j\omega})\right|$ versus ω

(b) Phase: $\angle H(e^{j\omega})$ versus ω

Ideal and nonideal filters

(1) Frequency domain specifications
(a) Passband
(b) Transition band
(c) Stop band
(d) Passband ripple
(e) Stopband ripple

(2) Time domain specifications
(a) Rise time
(b) Overshoot
(c) Ringing frequency
(d) Settling time

First-order continuous-time systems

(1) Differential equation: $\tau \dfrac{dy(t)}{dt} + y(t) = x(t)$

(a) τ is the time constant

(2) Frequency response: $H(j\omega) = \dfrac{1}{j\omega\tau + 1}$

(3) Impulse response: $h(t) = \dfrac{1}{\tau} e^{-t/\tau} u(t)$

(4) Step response: $s(t) = h(t) * u(t) = \left[1 - e^{-t/\tau}\right] u(t)$

(5) Bode plot

(a) $20\log_{10}|H(j\omega)| = -10\log_{10}\left[(\omega\tau)^2 + 1\right]$

(b) $\angle H(j\omega) = -\tan^{-1}(\omega\tau)$

Second-order continuous-time systems

(1) Differential equation: $\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t)$

(a) ζ is the damping ration

(b) ω_n is the undamped natural frequency

(2) Frequency response:

$$
\begin{aligned}
H(j\omega) &= \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} \\
&= \frac{\omega_n^2}{(j\omega - c_+)(j\omega - c_-)} \quad \text{where } c_\pm = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \\
&= \frac{M}{j\omega - c_+} - \frac{M}{j\omega - c_-} \quad \text{where } M = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}
\end{aligned}
$$

(3) Impulse response

(a) If $\zeta \neq 1$, $h(t) = M\left[e^{c_+ t} - e^{c_- t}\right]u(t)$

(a) If $\zeta = 1$, $h(t) = \omega_n^2\, t\, e^{-\omega_n t} u(t)$

(4) Step response

(a) If $\zeta \neq 1$, $s(t) = h(t) * u(t) = \left\{1 + M\left[\frac{e^{c_+ t}}{c_+} - \frac{e^{c_- t}}{c_-}\right]\right\} u(t)$

(a) If $\zeta = 1$, $s(t) = \left[1 - e^{-\omega_n t} - \omega_n\, t\, e^{-\omega_n t}\right] u(t)$

(5) Bode plot

(a) $20\log_{10}|H(j\omega)| = -10\log_{10}\left\{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right\}$

(b) $\angle H(j\omega) = -\tan^{-1}\left(\dfrac{2\zeta(\omega/\omega_n)}{1-(\omega/\omega_n)^2}\right)$

(6) Note that

(a) $\angle H(j\omega_n) = -\dfrac{\pi}{2}$

(b) If $\zeta < \sqrt{2}/2 \cong 0.707$, $\omega_{\max} = \omega_n\sqrt{1-2\zeta^2} = \arg\min_{\omega}|H(j\omega)|$

(c) If $\zeta > \sqrt{2}/2 \cong 0.707$, $|H(j\omega)|$ decreases monotonically

(d) Quality factor, $Q = \dfrac{1}{2\zeta}$ defines a measure of the sharpness of the peak

First-order discrete-time systems

(1) Difference equation: $y[n] - a\,y[n-1] = x[n]$, $|a| < 1$

(a) a determines the rate of the system response

(2) Frequency response: $H(e^{j\omega}) = \dfrac{1}{1-ae^{-j\omega}}$

(3) Impulse response: $h[n] = a^n u[n]$

(4) Step response: $s[n] = h[n] * u[n] = \dfrac{1-a^{n+1}}{1-a}u[n]$

(5) Bode plot

(a) $\left|H(e^{j\omega})\right| = \dfrac{1}{\left(1+a^2-2a\cos\omega\right)^{1/2}}$

(b) $\angle H(e^{j\omega}) = -\tan^{-1}\left[\dfrac{a\sin\omega}{1-a\cos\omega}\right]$

Second-order discrete-time systems

(1) Difference equation: $y[n] - 2r\cos\theta\, y[n-1] + r^2 y[n-2] = x[n]$, $0 < r < 1$ and $0 \le \theta \le \pi$

(a) r controlls the rate of decay of $h[n]$

(b) θ determines the frequency of oscillation

(2) Frequency response

$$H(e^{j\omega}) = \frac{1}{1-2r\cos\theta e^{-j\omega}+r^2e^{-j2\omega}} = \frac{1}{\left[1-\left(re^{j\theta}\right)e^{-j\omega}\right]\left[1-\left(re^{-j\theta}\right)e^{-j\omega}\right]}$$

(a) If $\theta \neq 0$ and $\theta \neq \pi$, $H(e^{j\omega}) = \frac{A}{\left[1-\left(re^{j\theta}\right)e^{-j\omega}\right]} + \frac{B}{\left[1-\left(re^{-j\theta}\right)e^{-j\omega}\right]}$ where

$A = \frac{e^{j\theta}}{2j\sin\theta}$ and $B = \frac{e^{-j\theta}}{2j\sin\theta}$

(b) If $\theta = 0$, $H(e^{j\omega}) = \frac{1}{\left(1-re^{-j\omega}\right)^2}$

(c) If $\theta = \pi$, $H(e^{j\omega}) = \frac{1}{\left(1+re^{-j\omega}\right)^2}$

(3) Impulse response

(a) If $\theta \neq 0$ and $\theta \neq \pi$, $h[n] = \left[A\left(re^{j\theta}\right)^n + B\left(re^{j\theta}\right)^n\right]u[n] = r^n\frac{\sin[(n+1)\theta]}{\sin\theta}u[n]$

(a) If $\theta = 0$, $h[n] = (n+1)r^n u[n]$

(c) If $\theta = \pi$, $h[n] = (n+1)(-r)^n u[n]$

(4) Step response

(a) If $\theta \neq 0$ and $\theta \neq \pi$, $s[n] = \left[A\left(\frac{1-\left(re^{j\theta}\right)^{n+1}}{1-re^{j\theta}}\right) + B\left(\frac{1-\left(re^{-j\theta}\right)^{n+1}}{1-re^{-j\theta}}\right)\right]u[n]$

(a) If $\theta = 0$, $s[n] = \left[\frac{1}{(r-1)^2} - \frac{r}{(r-1)^2}r^n + \frac{r}{r-1}(n+1)r^n\right]u[n]$

(c) If $\theta = \pi$, $s[n] = \left[\frac{1}{(r+1)^2} + \frac{r}{(r+1)^2} r^n + \frac{r}{r+1}(n+1)(-r)^n \right] u[n]$

(5) If A and B are real for the case where $\theta \neq 0$ and $\theta \neq \pi$,

$$H(e^{j\omega}) = \frac{1}{\left(1 - d_1 e^{-j\omega}\right)\left(1 - d_2 e^{-j\omega}\right)} \quad \text{with} \quad |d_1| < 1 \quad \text{and} \quad |d_2| < 1$$

It corresponds to the difference equation, $y[n] - (d_1 + d_2) y[n-1] + d_1 d_2 y[n-2] = x[n]$. Then, the frequency response is

$$H(e^{j\omega}) = \frac{A}{1 - d_1 e^{-j\omega}} + \frac{B}{1 - d_2 e^{-j\omega}}$$

where $A = \frac{d_1}{d_1 - d_2}$ and $B = \frac{d_2}{d_2 - d_1}$. Impulse response and step response are

$$h[n] = \left[A d_1^n + B d_2^n \right] u[n] \quad \text{and}$$

$$s[n] = \left[A\left(\frac{1 - d_1^{n+1}}{1 - d_1} \right) + B\left(\frac{1 - d_2^{n+1}}{1 - d_2} \right) \right] u[n].$$

10. Dualities in Fourier Transform

	Continuous time		**Discrete time**	
	Time domain	*Frequency domain*	*Time domain*	*Frequency domain*
Fourier series	continuous time periodic in time $x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	discrete frequency aperiodic in frequency $a_k=\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$	discrete time periodic in time $x[n]=\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	discrete frequency periodic in frequency $a_k=\frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}$
Fourier transform	continuous time aperiodic in time $x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$	continuous frequency aperiodic in frequency $X(j\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$	discrete time aperiodic in time $x[n]=\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$	continuous frequency periodic in frequency $X(e^{j\omega})=\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

11. 표본화와 표본화 주파수의 변환 (Sampling and Sampling Frequency Conversion)

표본화 (Sampling)

연속-시간 신호 $x_c(t)$를 T 시간 간격으로 표본화하는 경우를 생각해 보자. 이때, $x_c(t)$는 band-limited 신호로서 최대 주파수가 $\Omega_M = 2\pi f_M$이라 가정한다. 편의상 연속-시간 신호의 각주파수는 Ω로 나타내고 이산-시간 신호의 각주파수는 ω로 나타낸다. $x_c(t)$의 연속-시간 Fourier 변환쌍은 다음과 같다.

$$x_c(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_c(j\Omega)e^{j\Omega t}d\Omega \Leftrightarrow X_c(j\Omega) = \int_{-\infty}^{\infty} x_c(t)e^{-j\Omega t}dt$$

임펄스 열을 $x_c(t)$에 곱하는 것으로 표본화 과정을 모델링하고, 표본화 후의 신호를 $x_s(t)$라 하자.

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) \Leftrightarrow S(j\Omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta\left(\Omega - k\frac{2\pi}{T}\right) = \Omega_s \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)$$

$$x_s(t) = x_c(t)s(t) = x_c(t)\sum_{n=-\infty}^{\infty} \delta(t-nT) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t-nT)$$

이때, $\Omega_s = 2\pi f_s = \frac{2\pi}{T}$는 표본화 주파수이다. 그러면, $x_s(t)$의 연속-시간 Fourier 변환은 다음과 같이 두가지로 구할 수 있다.

$$X_s(j\Omega) = \frac{1}{2\pi}X_c(j\Omega) * S(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j\Omega - jk\Omega_s)$$

$$\begin{aligned} X_s(j\Omega) &= \int_{-\infty}^{\infty} x_s(t)e^{-j\Omega t}dt = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t-nT)e^{-j\Omega t}dt \\ &= \sum_{n=-\infty}^{\infty} x_c(nT)\int_{-\infty}^{\infty} \delta(t-nT)e^{-j\Omega t}dt = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-jn\Omega T} \end{aligned}$$

연속-시간 신호 $x_s(t)$에서 시간 축을 T로 나누어 정규화(normalize)하고,

임펄스 열을 이산-시간 수열(sequence)로 변환하여 얻어지는 이산-시간 신호를 $x[n]$이라 하면,

$$x[n] = x_c(nT)$$

이고, $x[n]$의 이산-시간 Fourier 변환쌍은 다음과 같다.

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega \quad \Leftrightarrow \quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

그런데, $X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-jn\Omega T} = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\Omega T}$ 이므로

$$X_s(j\Omega) = X(e^{j\omega})\Big|_{\omega=\Omega T} = X(e^{j\Omega T})$$

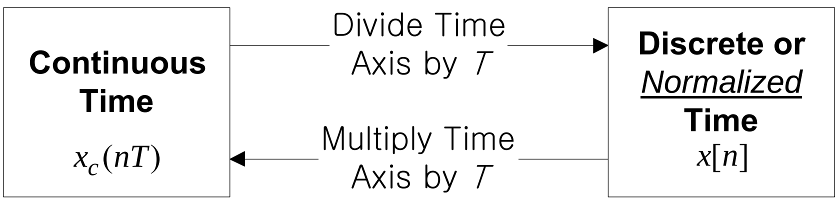
이다. 따라서, 두 각주파수 사이에는

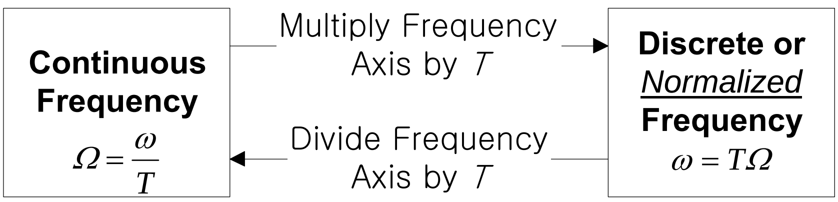
$$\omega = T\Omega$$

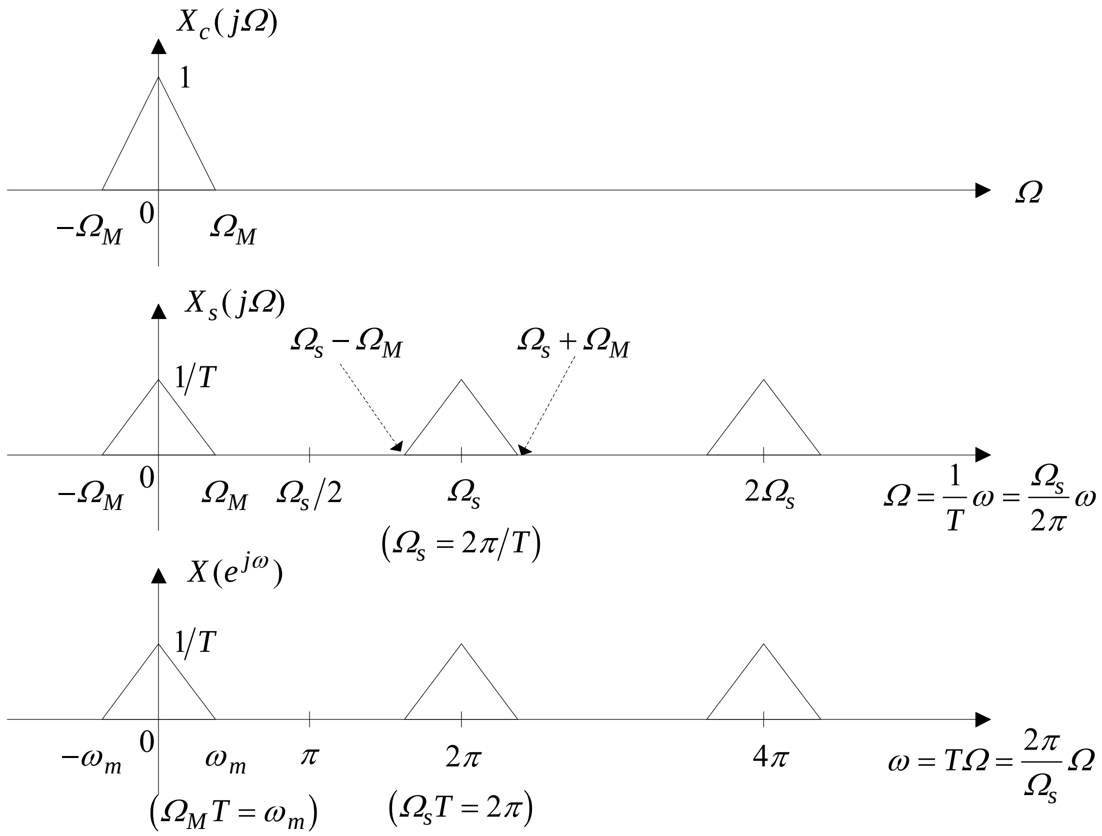
의 관계가 성립하고, 이산-시간 신호 $x[n]$의 Fourier 변환은 다음의 두가지로 나타낼 수 있다.

$$X(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j\Omega - jk\Omega_s)$$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right)$$







Nyquist 표본화 정리 (Nyquist sampling theorem)

표본화 주파수와 신호의 최대 주파수 사이에 다음의 관계가 성립하면, 이득이 T 이고 차단주파수가 Ω_M 인 이상적인 저역통과 필터를 이용하여 표본화하여 얻어진 이산-시간 신호 $x[n]$ 으로부터 원래의 연속-시간 신호 $x_c(t)$ 를 오차 없이 완벽하게 복원할 수 있다.

$$\Omega_s > 2\Omega_M \text{ 또는 } \Omega_M < \frac{\Omega_s}{2} \text{ 또는 } \omega_m < \pi$$

이때, $2\Omega_M$ 을 Nyquist rate 이라하고 $\frac{\Omega_s}{2}$ 를 Nyquist 주파수라고 한다.

연속-시간 신호의 복원 (Reconstruction of continuous-time signal)

$x[n]$을 이득이 T이고 차단주파수가 Ω_M 인 이상적인 저역통과 필터에 통과시키면 interpolation 에 의해 원래의 연속-시간 신호 $x_c(t)$ 가 복원된다. 표본

화 후의 신호 $x_s(t)=\sum_{n=-\infty}^{\infty}x_c(nT)\delta(t-nT)=\sum_{n=-\infty}^{\infty}x[n]\delta(t-nT)$ 를 이득이 T 이고 차단주파수가 $\pi/T=\Omega_s/2$ 인 이상적인 저역통과 필터에 통과시켰다고 가정해 보자.

$$h(t)=\frac{\sin\pi t/T}{\pi t/T} \Leftrightarrow H(j\Omega)=\begin{cases}T, & |\Omega|\le\pi/T\\ 0, & \text{otherwise}\end{cases}$$

이므로, 저역통과 필터의 출력은

$$\tilde{x}(t)=x_s(t)*h(t)=\sum_{n=-\infty}^{\infty}x[n]\frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

가 되며, $x[n]$들이 sinc 함수에 의해 interpolation 되어서 $\tilde{x}(t)=x_c(t)$ 가 된다.

표본화 주파수 변환 (Sampling frequency conversion)

(1) 정수배의 downsampling

표본화 주기 T 로 표본화한 신호 $x[n]$ 으로부터, 표본화 주기 $T'=MT$ 로 표본화한 신호 $x_M[n]$을 구하는 경우를 생각해 보자.

$$x_M[n]=x[Mn]=x_c(nMT)=x_c(nT')$$

이고,

$$X(e^{j\omega})=\frac{1}{T}\sum_{k=-\infty}^{\infty}X_c\left(j\frac{\omega}{T}-j\frac{2\pi k}{T}\right)$$

이므로,

$$X_M(e^{j\omega'})=\frac{1}{MT}\sum_{r=-\infty}^{\infty}X_c\left(j\frac{\omega'}{MT}-j\frac{2\pi r}{MT}\right)$$

이고, $\omega=\frac{1}{M}\omega'$ 이다.

이제 $r=i+kM$ 으로 하면, $-\infty<k<\infty$ 이고 $0\le i\le(M-1)$ 이고,

$$X_M(e^{j\omega'})=\frac{1}{M}\sum_{i=0}^{M-1}\left\{\frac{1}{T}\sum_{k=-\infty}^{\infty}X_c\left(j\frac{\omega'}{MT}-j\frac{2\pi k}{T}-j\frac{2\pi i}{MT}\right)\right\}$$

이다. 위의 $X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right)$에서

$$X\left(e^{j(\omega'-2\pi i)/M}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega'-2\pi i}{MT} - j\frac{2\pi k}{T}\right)$$

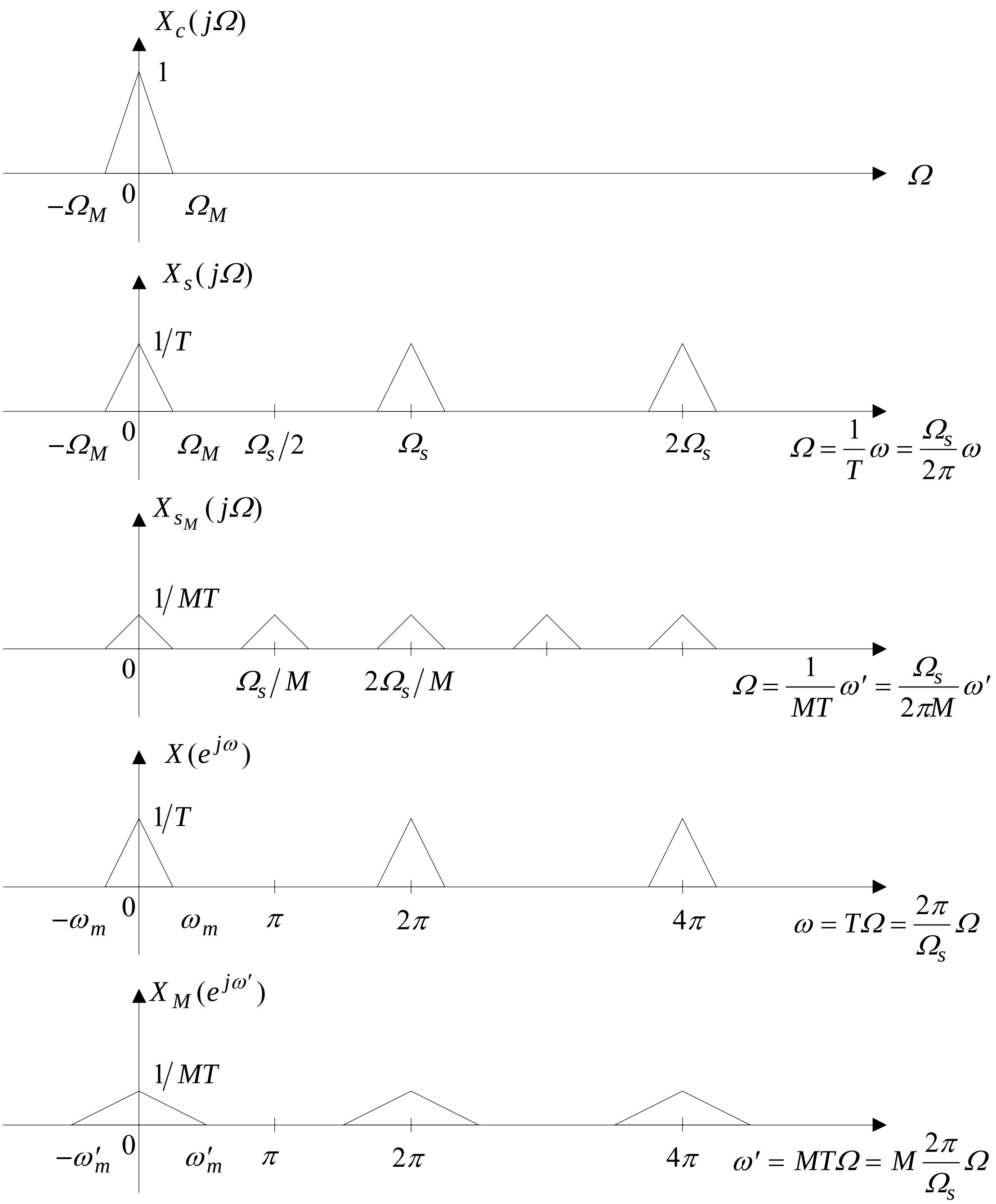
이므로,

$$X_M(e^{j\omega'}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j(\omega'/M-2\pi i/M)}\right)$$

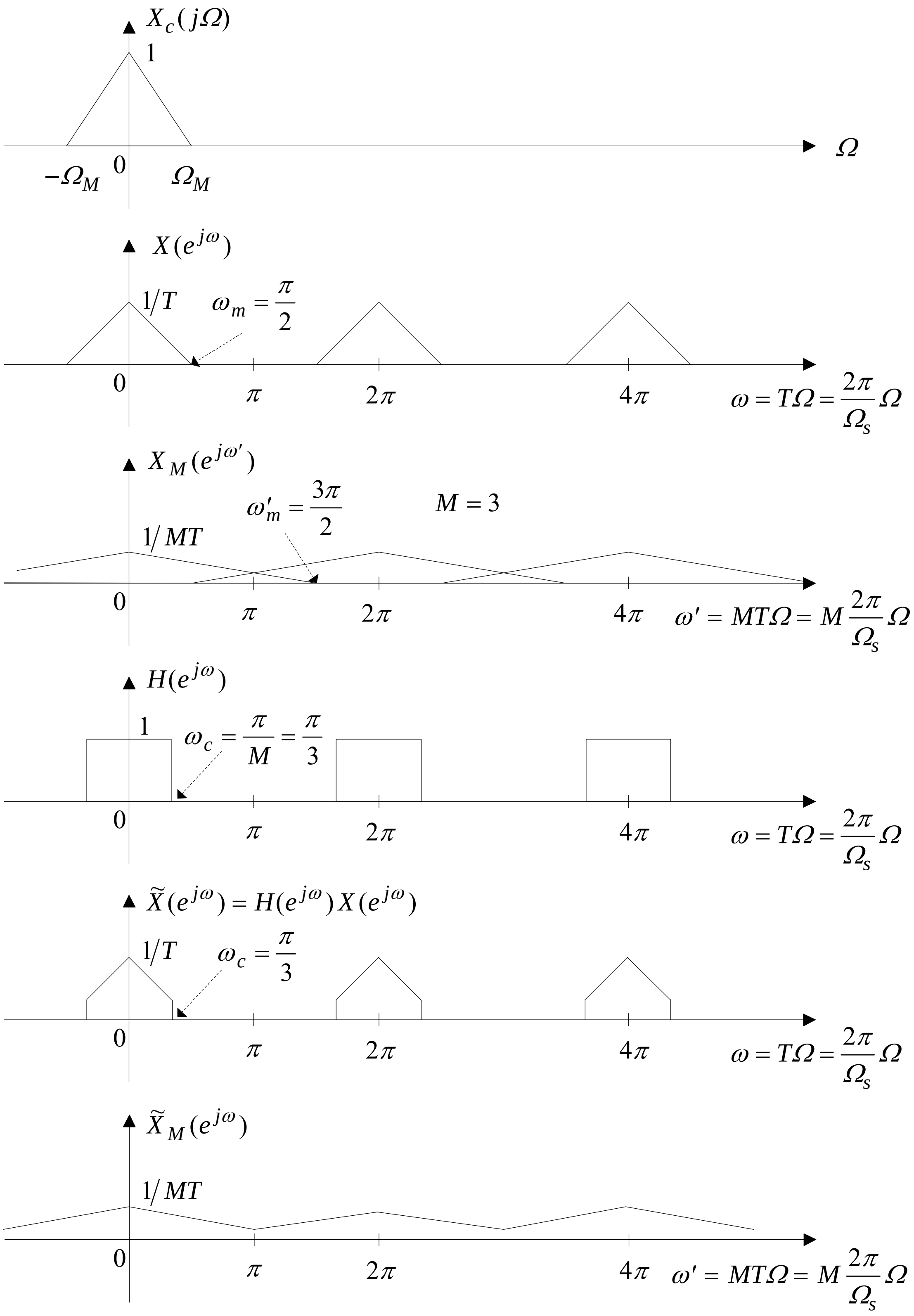
가 된다. Aliasing 이 발생하지 않기 위해서는

$\omega'_m = MT\Omega_M < \pi$ 또는 $\frac{2\pi}{M} > 2T\Omega_M = 2\omega_m$ 또는 $\frac{\pi}{M} > \omega_m$ 또는 $\Omega_s > 2M\Omega_M$

이어야 한다.



최초의 표본화 주파수가 충분히 크지 않아서 (즉, $\Omega_s < 2M\Omega_M$), M-downsampling 에 의해 aliasing 이 발생할 수 밖에 없는 경우에는, 차단주파수가 π/M 인 이상적인 디지털 저역통과 필터를 사용한 후, downsampling 한다. 이러한 과정을 decimation 이라 한다.


$X_c(j\Omega)$
1
Ω
$-\Omega_M$
0
Ω_M
$X(e^{j\omega})$
$1/T$
$\omega_m = \frac{\pi}{2}$
0
π
2π
4π
$\omega = T\Omega = \frac{2\pi}{\Omega_s}\Omega$
$X_M(e^{j\omega'})$
$\omega'_m = \frac{3\pi}{2}$
$M = 3$
$1/MT$
0
π
2π
4π
$\omega' = MT\Omega = M\frac{2\pi}{\Omega_s}\Omega$
$H(e^{j\omega})$
1
$\omega_c = \frac{\pi}{M} = \frac{\pi}{3}$
0
π
2π
4π
$\omega = T\Omega = \frac{2\pi}{\Omega_s}\Omega$
$\tilde{X}(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$
$1/T$
$\omega_c = \frac{\pi}{3}$
0
π
2π
4π
$\omega = T\Omega = \frac{2\pi}{\Omega_s}\Omega$
$\tilde{X}_M(e^{j\omega})$
$1/MT$
0
π
2π
4π
$\omega' = MT\Omega = M\frac{2\pi}{\Omega_s}\Omega$

(2) 정수배의 upsampling

표본화 주기 T 로 표본화한 신호 $x[n]$으로부터, 표본화 주기 $T'=T/L$ 로 표본화한 신호 $x_L[n]$을 구하는 경우를 생각해 보자. n 이 L 의 배수인 경우에는 $x_L[n]=x[n/L]=x_c(nT/L)=x_c(nT')$ 이고, n 이 L 의 배수가 아닌 경우에는 $x_L[n]=0$으로 한다. 즉,

$$x_L[n]=\begin{cases} x[n/L], & n=0,\pm L,\pm 2L,\cdots \\ 0, & \text{otherwise} \end{cases}$$

또는,

$$x_L[n]=\sum_{k=-\infty}^{\infty} x[k]\delta[n-kL]$$

이다. 이산-시간 Fourier 변환을 하면

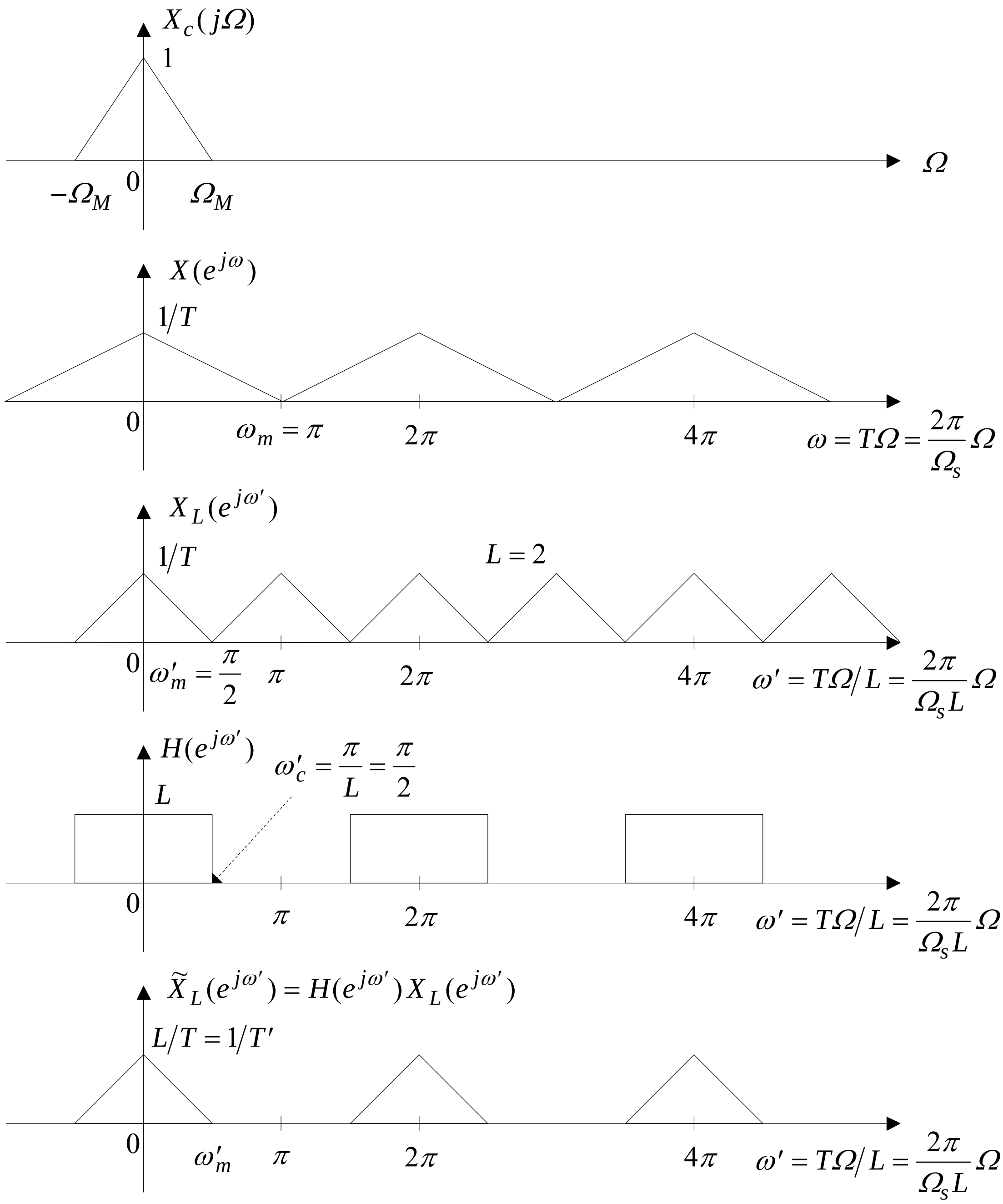
$$X_L(e^{j\omega'})=\sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} x[k]\delta[n-kL]\right)e^{-j\omega' n}=\sum_{k=-\infty}^{\infty} x[k]e^{-j\omega' Lk}=X(e^{j\omega' L})$$

따라서, $X_L(e^{j\omega'})$ 은 $X(e^{j\omega})$ 의 ω 축을 L 로 나누어줌으로써 구해진다 (즉, $\omega=L\omega'$). 이것은 ω 축을 L 배 만큼 압축함을 의미한다. 연속-시간 각주파수와의 관계는

$$\omega'=\Omega T'=\frac{\Omega T}{L}$$

이다. 따라서, upsampling 은 그 자체에 의한 aliasing 이 발생하지 않는다.

원래의 신호 $x[n]$이 충분히 큰 표본화 주파수에 의해 표본화된 신호여서 aliasing 이 발생하지 않았다고 가정하자. 이득이 L 이고 차단주파수가 π/L 인 이상적인 디지털 저역통과 필터에 $x_L[n]$을 통과시키면, interpolation 에 의해 그 출력 $\tilde{x}_L[n]$은 원래의 연속-시간 신호 $x_c(t)$ 를 $T'=T/L$ 의 표본화 주기로 빠르게 표본화한 결과와 동일하게 된다.



(3) 유리수배의 표본화 주파수 변환

L-interpolator 의 출력을 M-decimator 의 입력에 연결하는 cascade 결합에 의해 M/L 배의 표본화 주파수를 얻을 수 있다.

12. LTI 시스템의 주파수 영역 해석 (Analysis of LTI System in Frequency Domain)

LTI (Linear Time Invariant) system

LTI 시스템의 입출력 관계는 다음과 같다.

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \quad \overset{Z}{\longleftrightarrow} \quad Y(z) = H(z)X(z)$$

주파수 응답 (frequency response)은 $z = e^{j\omega}$ 에서, $Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$ 이고

따라서 $H(e^{j\omega}) = \dfrac{Y(e^{j\omega})}{X(e^{j\omega})}$ 이다. 이때,

(1) $\left|H(e^{j\omega})\right| = \dfrac{\left|Y(e^{j\omega})\right|}{\left|X(e^{j\omega})\right|}$: magnitude response 또는 gain

(2) $\arg\left\{H(e^{j\omega})\right\} = \angle H(e^{j\omega}) = \angle Y(e^{j\omega}) - \angle X(e^{j\omega})$: phase response 또는 phase shift

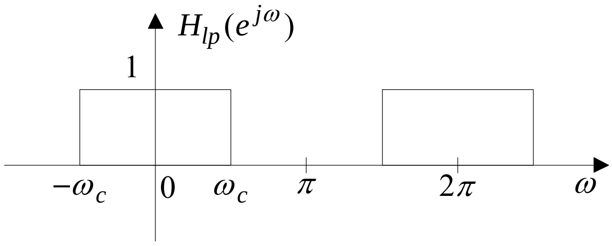
(3) $\tau(\omega) = \mathrm{grd}\left\{\angle H(e^{j\omega})\right\} = -\dfrac{d}{d\omega}\left\{\angle H(e^{j\omega})\right\}$: group delay

여기에서, group delay 는 “number of samples”을 그 단위로 가지며, 위상의 선형성을 나타내는 척도이다. 만약 group delay 가 상수가 아니라면 위상지연이 비선형적임을 의미한다.

이상적인 LPF 및 HPF

이상적인 LPF 는 다음과 같은 주파수 응답을 가진다. 즉,

$$H_{lp}(e^{j\omega}) = \begin{cases} 1, & \omega \le |\omega_c| \\ 0, & \omega_c < |\omega| < \pi \end{cases}$$



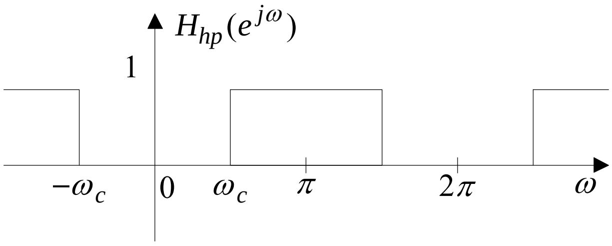
따라서,

$$\begin{aligned} h_{lp}[n] &= \frac{1}{2\pi}\int_{2\pi} H_{lp}(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega \\ &= \begin{cases} \dfrac{\omega_c}{\pi}, & n=0 \\ \dfrac{1}{2\pi jn}e^{j\omega n}\Big|_{-\omega_c}^{\omega_c} = \dfrac{\sin\omega_c n}{\pi n}, & n\neq 0 \end{cases} \\ &= \frac{\sin\omega_c n}{\pi n} \end{aligned}$$

이고, 이것은 noncausal system 이다.

이상적인 HPF 는 다음과 같은 주파수 응답을 가진다. 즉,

$$H_{hp}(e^{j\omega}) = 1 - H_{lp}(e^{j\omega})$$



따라서, $h_{hp}[n] = \delta[n] - h_{lp}[n] = \delta[n] - \dfrac{\sin\omega_c n}{\pi n}$ 이고, 역시 noncausal system 이다.

이상적인 지연 (delay)

입력신호를 n_d 샘플 만큼 지연시키기만 하는 시스템을 고려해 보자.

$h_d[n] = \delta[n-n_d]$이고, $H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty}\delta[n-n_d]e^{-j\omega n} = e^{-j\omega n_d}$ 이다. 따라서,

$$\left|H_d(e^{j\omega})\right| = 1 \ \text{이고}\ \angle H_d(e^{j\omega}) = -\omega n_d,\ -\pi < \omega \le \pi$$

이다. 위상지연이 주파수의 선형함수임을 주목하라. 이를 linear phase 라고 함.

Linear phase 특성을 가지는 이상적인 LPF

앞의 이상적인 LPF 가 n_d 샘플 만큼의 이상적인 지연을 가진다면,

$$H_{lp}(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & \omega \le |\omega_c| \\ 0, & \omega_c < |\omega| < \pi \end{cases} \quad \text{그리고} \quad h_{lp}[n] = \frac{\sin\omega_c(n-n_d)}{\pi(n-n_d)}$$

이다. 이 역시 noncausal system 이다.

Linear constant coefficient difference equation 으로 표현되는 LTI 시스템

이에 속하는 LTI 시스템은 다음의 식으로 표현된다.

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \quad \overset{Z}{\longleftrightarrow} \quad \sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

따라서, 전달함수는 rational function 이 되고

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}\left(1-c_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1-d_k z^{-1}\right)}$$

이며, 우리는 다음의 사항들을 알 수 있다.

(1) 분자의 $\left(1-c_k z^{-1}\right)$ 은 zero at $z=c_k$ 와 pole at $z=0$ 를 의미한다.

(2) 분모의 $\left(1-d_k z^{-1}\right)$ 은 pole at $z=d_k$ 와 zero at $z=0$ 를 의미한다.

(3) $h[n]=0$ for $n<0$ 즉, 시스템이 causal 하다면, ROC 는 가장 밖에 위치한 pole 의 바깥쪽이 된다.

(4) 시스템이 stable 하기 위해서는 ROC 가 반드시 단위원 (unit circle) 즉 $|z|=1$ 을 포함하여야 한다.

(5) 시스템이 causal 하고 또 stable 하기 위해서는 모든 pole 들은 단위원 내부에 위치하여야 한다.

Inverse 시스템

전달함수 $H(z)$ 를 가지는 LTI 시스템의 inverse 시스템을 고려해 보자. 즉, $G(z)=H(z)H_i(z)=1$ 또는 $g[n]=h[n]*h_i[n]=\delta[n]$ 이어야 하므로, inverse 시스템의 전달함수는

$$H_i(z) = \frac{1}{H(z)} \quad \text{또는} \quad H_i(e^{j\omega}) = \frac{1}{H(e^{j\omega})}$$

이다. 만약, $H_i(z)$ 가 존재한다면, 이는 $H(z)$ 에 의한 효과를 완벽하게 상쇄할 수 있음을 의미한다. 그러나, 어떤 ω에서 $H(e^{j\omega}) = 0$ 이라면, inverse 시스템은 존재하지 않게되고, 이는 한번 0 이 되면 되살릴 수 있는 방법이 없음을 의미한다. 이때, $H(z)$ 의 zero 의 위치와 $H(e^{j\omega}) = 0$ 의 관계는?

만약, inverse 시스템, $H_I(z)$ 이 존재하고 $H(z)$ 와 $H_i(z)$ 가 모두 causal 및 stable 하다면, 이는 $H(z)$ 의 모든 pole 들과 모든 zero 들이 단위원 내부에 위치하여야 함을 의미한다. 이와같은 시스템을 *minimum phase system* 이라고 한다.

Impulse response: IIR and FIR

전달함수 $H(z)$ 의 pole 들은 모두 single pole 이라면, 다음과 같이 전개할 수 있다. 즉,

$$X(z) = \underbrace{\sum_{r=0}^{M-N} B_r z^{-r}}_{\text{if } M>N} + \underbrace{\sum_{k=1,k\neq i}^{N} \frac{A_k}{1-d_k z^{-1}}}_{\text{single poles}}$$

따라서, impulse response 는

$$h[n] = \sum_{r=0}^{M-N} B_r \delta[n-r] + \sum_{k=1}^{N} A_k d_k^n u[n]$$

이다. 이때, 모든 A_k 들이 다 0 이면 $h[n]$ 의 길이가 유한하고, 그렇지 않으면 $h[n]$ 은 무한한 길이를 가진다. 따라서,

(1) FIR (finite impulse response): $A_k = 0, \ \forall k$, no pole, all zero

예: $h[n] = \begin{cases} a^n, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$: FIR

(2) IIR (infinite impulse response): $A_k \neq 0$, for some k, at least one pole

예: $y[n] - ay[n-1] = x[n] \ \Rightarrow \ h[n] = a^n u[n]$: IIR

Frequency response of rational system function

(1) Gain in dB = $20\log\left|H(e^{j\omega})\right|$

- Attenuation in dB = $-20\log\left|H(e^{j\omega})\right|$ = – (Gain in dB)

(2) $\arg\left\{H(e^{j\omega})\right\} = \angle H(e^{j\omega}) = \angle Y(e^{j\omega}) - \angle X(e^{j\omega})$

- Principal value 로 계산되며, 따라서 $-\pi < \angle H(e^{j\omega}) \le \pi$ 의 값을 가진다.
- 따라서, $\pm 2\pi$ 의 discontinuity 가 발생할 수 있다.
- 이러한 $\pm 2\pi$ discontinuity 를 모두 제거하여 연속함수로 만드는 것은 phase unwrapping 이라 한다.

(3) $\tau(\omega) = \text{grd}\left\{\angle H(e^{j\omega})\right\} = -\dfrac{d}{d\omega}\left\{\angle H(e^{j\omega})\right\}$

(4) Pole-zero plot 으로부터 frequency response 를 추론하는 것이 가능하다. 즉, 단위원을 돌면서
- Pole 이 가까워지면 gain 이 증가하고
- Zero 가 가까워지면 gain 이 감소한다.

(5) 예:

- $H(z) = \dfrac{z-a}{z}$, $a = re^{j\theta}$: $r>1, r=1, r<1$ 및 $\theta = 0, \theta = \dfrac{\pi}{2}, \theta = \pi$

- $H(z) = \dfrac{z^2}{(z-a)^2}$, $a = re^{j\theta}$: $r>1, r=1, r<1$ 및 $\theta = 0, \theta = \dfrac{\pi}{2}, \theta = \pi$

- $H(z) = \dfrac{0.05634\left(1+z^{-1}\right)\left(1-1.0116z^{-1}+z^{-2}\right)}{\left(1-0.683z^{-1}\right)\left(1-1.4461z^{-1}+0.7957z^{-2}\right)}$

$|H|$와 $\angle H$ 사이의 관계

Linear constant coefficient difference equation 으로 표현이 가능한 LTI 시스템의 전달함수 $H(z)$ 는 rational function 이다. 이때, $|H| = \left|H(e^{j\omega})\right|$ 와 $\angle H = \angle H(e^{j\omega})$ 사이에는 다음의 관계들이 성립한다.

(1) $\left.\begin{array}{l}|H| \text{ and} \\ \#\text{ of poles and zeros}\end{array}\right\} \Rightarrow$ finite number of choices for $\angle H$

(2) $|H|$ within a scale factor $\Leftarrow \left\{\begin{array}{l}\angle H \text{ and} \\ \#\text{ of poles and zeros}\end{array}\right.$

(3) If $H(z)$ is minimum phase, then

$$|H| \Rightarrow \angle H \text{ and } \alpha|H| \Leftarrow \angle H \text{ with } \alpha \text{ unknown.}$$

(4) Pole 과 zero 들의 관계

$$\left|H(e^{j\omega})\right|^2 = H(e^{j\omega})H^*(e^{j\omega}) = H(z)H^*(1/z^*)\Big|_{z=e^{j\omega}} = C(z)\Big|_{z=e^{j\omega}}$$

$$z = re^{j\omega},\ z^* = re^{-j\omega},\ \frac{1}{z} = \frac{1}{r}e^{-j\omega},\ \frac{1}{z^*} = \frac{1}{r}e^{j\omega}$$

$$H(z) = \frac{b_0}{a_0}\frac{\prod_{k=1}^{M}\left(1-c_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1-d_k z^{-1}\right)} \text{ and } H^*\!\left(\frac{1}{z^*}\right) = \frac{b_0}{a_0}\frac{\prod_{k=1}^{M}\left(1-c_k^* z\right)}{\prod_{k=1}^{N}\left(1-d_k^* z\right)}$$

$$C(z) = H(z)H^*\!\left(\frac{1}{z^*}\right) = \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M}\left(1-c_k z^{-1}\right)\left(1-c_k^* z\right)}{\prod_{k=1}^{N}\left(1-d_k z^{-1}\right)\left(1-d_k^* z\right)}$$

따라서, $C(z)$ 의 pole 과 zero 들은 각각 real 이거나 아니면 conjugate reciprocal pair 이어야 한다. 예를 들면, 다음의 두 시스템에서 $|H_1| = |H_2|$ 이나 $\angle H_1 \neq \angle H_2$ 이다. 즉,

$$H_1(z) = \frac{2\left(1-z^{-1}\right)\left(1+0.5z^{-1}\right)}{\left(1-0.8e^{j\pi/4}z^{-1}\right)\left(1-0.8e^{-j\pi/4}z^{-1}\right)},\quad H_2(z) = \frac{\left(1-z^{-1}\right)\left(1+2z^{-1}\right)}{\left(1-0.8e^{j\pi/4}z^{-1}\right)\left(1-0.8e^{-j\pi/4}z^{-1}\right)}$$

(5) Uncertainty: 만약, $H(z) = H_1(z)\dfrac{z^{-1}-a^*}{1-az^{-1}}$ 이라면

$$C(z) = H(z)H^*(1/z^*) = H_1(z)H_1^*(1/z^*)$$

이므로 불확실성이 존재한다. 따라서, all-pass system 을 분석해보자.

All-pass system

다음과 같은 전달함수를 가지는 시스템을 all-pass 시스템이라고 한다.

$$H_{ap}(z) = \frac{z^{-1}-a^*}{1-az^{-1}},\ |a| < 1,\ |z| > |a|$$

$$H_{ap}(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}} = e^{-j\omega}\frac{1 - a^* e^{j\omega}}{1 - ae^{-j\omega}} \quad \Rightarrow \quad \left|H_{ap}(e^{j\omega})\right| = 1$$

일반적인 all-pass 시스템의 전달함수는

$$H_{ap}(z) = \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{\left(z^{-1} - e_k^*\right)\left(z^{-1} - e_k\right)}{\left(1 - e_k z^{-1}\right)\left(1 - e_k^* z^{-1}\right)}$$

이고, $|d_k| < 1$ 및 $|e_k| < 1$일 때 stable 하고 causal 하게 된다. Pole 과 zero 는 각각 $M = (2M_c + M_r)$개 씩이고, 다음과 같은 특성을 가진다.

(1) $\left|H_{ap}(e^{j\omega})\right| = 1$

(2) $\angle H_{ap}(e^{j\omega}) \le 0$

(3) $\tau(\omega) \ge 0$

(4) Minimum phase system

따라서, phase 또는 group delay compensation 에 이용된다.

Minimum phase system

Minimum phase system 은 다음과 같은 특징을 가진다.

(1) 모든 pole 과 zero 는 단위원 내부에 존재한다.

(2) Inverse system 이 존재한다.

(3) Minimum phase 이다.

(4) $|H| \Rightarrow \angle H$ 그리고 $\alpha|H| \Leftarrow \angle H$ with α unknown

전달함수가 rational function 인 모든 LTI 시스템의 전달함수는 minimum phase 전달함수와 all-pass 전달함수의 곱으로 표현할 수 있다. 즉,

$$H(z) = H_{mp}(z)H_{ap}(z)$$

만약, $H(z)$가 단위원 밖에 하나의 zero 를 가지고 나머지의 모든 pole 과 zero 는 단위원 내부에 있는 경우를 가정해 보자. 그러면, $|c| < 1$에 대하여

$$H(z) = H_1(z)\left(z^{-1} - c^*\right) = H_1(z)\left(z^{-1} - c^*\right)\frac{z^{-1} - c^*}{1 - cz^{-1}} = H_{mp}(z)H_{ap}(z)$$

이다.

위와 같은 경우에, $H(z)$ 의 stable 하고 causal 한 inverse system 은

$H_i(z) = \dfrac{1}{H_{mp}(z)}$ 가 된다. 이때,

$$G(z) = H(z)H_i(z) = H_{ap}(z)$$

이고, $\left|G(e^{j\omega})\right| = 1$ 이며 $\angle G(e^{j\omega}) = \angle H_{ap}(e^{j\omega})$ 이다.

예제: 다음 시스템의 inverse system 은? Inverse filtering 한 후의 결과는?

$$H(z) = \left(1-0.9e^{j0.6\pi}z^{-1}\right)\left(1-0.9e^{-j0.6\pi}z^{-1}\right)\left(1-1.25e^{j0.8\pi}z^{-1}\right)\left(1-1.25e^{-j0.8\pi}z^{-1}\right)$$

Minimum phase system 의 주요 특성

(1) Minimum phase lag:

$$\angle H = \angle H_{mp} + \angle H_{ap} \quad \text{및} \quad \angle H_{ap} \le 0 \;\Rightarrow\; \angle H \le \angle H_{mp}$$

단, $\angle H(e^{j0}) = 0$ 를 위해서는 $H(e^{j0}) = \sum h[n] > 0$ 의 조건이 필요하다.

(2) Minimum group delay:

$$\text{grd}\{H\} = \text{grd}\{H_{mp}\} + \text{grd}\{H_{ap}\} \quad \text{및} \quad \text{grd}\{H_{ap}\} \ge 0 \;\Rightarrow\; \text{grd}\{H\} \ge \text{grd}\{H_{mp}\}$$

(3) Minimum energy delay:

$$|H| = |H_{mp}| \;\Rightarrow\; |h[0]| < |h_{mp}[0]|$$

$$\sum_{n=0}^{\infty} |h[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H(e^{j\omega})\right|^2 d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H_{mp}(e^{j\omega})\right|^2 d\omega = \sum_{n=0}^{\infty} |h_{mp}[n]|^2$$

다음과 같이 partial energy 를 정의하면

$$E[n] = \sum_{m=0}^{n} |h[m]|^2$$

이고

$$\sum_{m=0}^{n} |h[m]|^2 \le \sum_{m=0}^{n} |h_{mp}[m]|^2$$

따라서, 대부분의 에너지가 $n = 0$ 근처에 집중되어 있음을 알 수 있다.

참고로 maximum phase system 은 모든 zero 가 단위원 외부에 존재하고,

maximum energy delay 의 특성을 가진다.